Name: SOLUTIONS.

Number Theory for Teachers–Practice Exam 2

For full credit, all work must be shown and clearly presented. No calculators.

1	
2	
3	
4	
5	

1. Using that 3 is a generator of U_{31}, make a table of logarithms. Use your table of logarithms to solve the following congruence:

$$2x^3 \equiv 4 \mod 31$$

(10 points)

$3^1 = 3$
$3^2 = 9$
$3^3 = 27 = -4$
$3^4 = -12 = 19$
$3^5 = -36 = -5 = 26$
$3^6 = -15 = 16$
$3^7 = 48 = 17$
$3^8 = 51 = 20$
$3^9 = 60 = 29 = -2$
$3^{10} = -6 = 25$
$3^{11} = -18 = 13$
$3^{12} = 39 = 8$
$3^{13} = 24 = -7$
$3^{14} = -21 = 10$
$3^{15} = 30 = -1$
$3^{16} = -3 = 28$
$3^{17} = -9 = 22$
$3^{18} = -27 = 4$
$3^{19} = 12$
$3^{20} = 36 = 5$
$3^{21} = 15$
$3^{22} = 45 = 14$
$3^{23} = 42 = 11$
$3^{24} = 33 = 2$
$3^{25} = 6$
$3^{26} = 18$
$3^{27} = 54 = 23 = -8$
$3^{28} = -24 = 7$
$3^{29} = 21$
$3^{30} = 63 = 1$

n	1	2	3	4	5	6	7	8	9	10	11	12
$\log_3 n$	30	24	1	18	20	25	28	12	2	14	23	19

n	13	14	15	16	17	18	19	20	21	22
$\log_3 n$	11	22	21	6	7	26	4	8	29	17

n	23	24	25	26	27	28	29	30
$\log_3 n$	27	13	10	5	3	16	9	15

$$\log_3 2x^3 \equiv \log_3 4 \mod 30$$

$$\log_3 2 + 3\log_3 x \equiv \log_3 4 \mod 30$$

$$24 + 3\log_3 x \equiv 18 \mod 30$$

$$3\log_3 x = -6 + 30k$$

$$\log_3 x = -2 + 10k$$
$$= 8, 18, 28, \ldots$$

So $x \equiv 3^8, 3^{18}, 3^{28} \mod 31$

and the solutions are

$$\boxed{x \equiv 20, \quad x \equiv 4 \text{ and } x \equiv 7 \mod 31}$$

4. Compute 3^{3^3} mod 10. (Hint: To do this computation, you should consider the exponent mod what number?) (10 points)

For 3^{3^3} mod 10, what matters is the exponent mod $\varphi(10) = \varphi(2)\cdot\varphi(5) = 1\cdot 4 = 4$.

So consider the exponent, 3^3 mod 4.

27 mod 4 is 3.

So $3^{3^3} = 3^{27} \equiv 3^3 \mod 10$

and $3^{3^3} \equiv 27 \mod 10$

$\equiv 7 \mod 10$

5. Clearly state and prove a divisibility test for 11. (10 points)

To test whether a number is divisible by 11, compute an alternating sum of its digits, i.e. (ones digit) − (tens digit) + (hundreds digit) + ... etc.

Algebraically, if our original number is

$a_n 10^n + \cdots + a_1 10 + a_0$ where the a_i are the digits,

we compute $a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n$.

If this alternating sum is divisible by 11, then so is the original number. If the alternating sum is not divisible by 11, then neither is the original number.

This test works because the alternating sum is congruent to the original number mod 11.

Since $10 \equiv -1 \mod 11$, $10^2 \equiv 1 \mod 11$, $10^3 \equiv -1 \mod 11$ etc,

$a_n 10^n + \cdots + a_1 \cdot 10^1 + a_0 \equiv a_0 + (-1) a_1 + a_2 - a_3 + \cdots + (-1)^n a_n \mod 11$.

Thus, the alternating sum is 0 mod 11 (i.e. is divisible by 11) if and only if the original number is 0 mod 11 (i.e. is divisible by 11).

2. How many 10th roots of 1 are there mod 151? (Note: 151 is prime. You do not need to find the roots.) Explain your answer. (10 points)

We know there are 10^{th} roots of 1 mod 151, e.g. $1^{10} \equiv 1 \bmod 151$.
If g is a generator mod 151, we can use logarithms base g to solve

$$x^{10} \equiv 1 \bmod 151.$$

Namely, we will have

$$10 \log_g x \equiv \log_g 1 \bmod 150$$

Now we have an equation of the form $10k \equiv b \bmod m$.
We know this has $(10, m)$ solutions if there are any solutions.
So there are $(10, 150) = 10$ solutions for $\log_g x$, and
therefore $\boxed{10 \text{ solutions}}$ x to $x^{10} \equiv 1 \bmod 151$.

3. How many solutions are there to the following congruences? You do not need to prove your answer, but show all relevant computations. (5 points each)

(a) $15x \equiv 20 \mod 58$

(b) $15x \equiv 20 \mod 57$

(c) $15x \equiv 21 \mod 60$

(d) $6307x \equiv 21 \mod 6853$

a) $(15, 58) = 1$, so there is one solution

b) $(15, 57) = 3$ and $3 \nmid 20$, so there are no solutions

c) $(15, 60) = 15$ and $15 \nmid 21$, so there are no solutions

d) Find $(6853, 6307)$:

$$6853 = 1 \cdot 6307 + 546$$
$$6307 = 11 \cdot 546 + 301$$
$$546 = 1 \cdot 301 + 245$$
$$301 = 1 \cdot 245 + 56$$
$$245 = 4 \cdot 56 + 21$$
$$56 = 2 \cdot 21 + 14$$
$$21 = 1 \cdot 14 + \boxed{7}$$
$$14 = 2 \cdot 7 + 0$$

$(6853, 6307) = 7$ and

$7 \mid 21$, so

there are 7 solutions

$$\begin{array}{r} 546 \\ \times\ 11 \\ \hline 5460 \\ 546 \\ \hline 6006 \end{array} \qquad \begin{array}{r} 56 \\ \times\ 4 \\ \hline 224 \end{array}$$