Math 290-Number Theory for Teachers
Problem of the Day #5

Today, you will begin to discover the secrets of the Magic Box!

Here is a first example. First, this is a *simple continued fraction.* (Simple because all the numerators are 1's. You may want to check that this fraction equality is true before going on.)

$$\frac{52}{35} = 1 + \cfrac{1}{2 + \cfrac{1}{17}}$$

The *convergents* of this continued fraction are

$$1,\ 1 + \frac{1}{2},\ \text{and } 1 + \cfrac{1}{2 + \cfrac{1}{17}}.$$

They are called convergents because they converge to the final answer of $\frac{52}{35}$.

Next, an "empty" magic box.

		1	2	17
0	1			
1	0			

Now, the magic box filled.

		1	2	17
0	1	1	3	52
1	0	1	2	35

1. Describe as many patterns as you can. Here are some suggestions.

- Compare the entries in the magic box with the convergents of the continued fractions.
- Can you fill in an "empty" magic box without computing convergents? (Hint: The two rows you fill in are independent of each other, so you could fill in the entire first row without doing the bottom row, or the entire bottom row without doing the top row. Find the pattern for each row.)
- What can you say about the 2×2 determinants? (In the above example, the determinants are $\left|\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right| = 0 \cdot 0 - 1 \cdot 1$, $\left|\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right| = 1 \cdot 1 - 0 \cdot 1$, $\left|\begin{smallmatrix} 1 & 3 \\ 1 & 2 \end{smallmatrix}\right| = 1 \cdot 2 - 3 \cdot 1$, and so on.)
- How might that help you find x and y such that $52x + 35y = 1$? $ax + by$?

To help your magic box quest, here are some more examples of filled magic boxes.

		1	2	1	1	3	1
0	1	1	3	4	7	25	32
1	0	1	2	3	5	18	23

		2	1	1	1	3
0	1	2	3	5	8	29
1	0	1	1	2	3	11