Math 290-Number Theory for Teachers
Problem of the Day #14
Due Wednesday, March 12, 2014

For today's problem of the day, you'll investigate the divisibility rules for 11 and 7. When I write $a_n a_{n-1} ... a_2 a_1 a_0$, I mean the number $a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \ldots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$. (That is, the number with a_n, a_{n-1}, etc. as digits.)

1. The divisibility rule for 11 says: to determine whether a number $a_n a_{n-1} ... a_2 a_1$ is divisible by 11, compute the alternating sum of the digits, i.e. $a_n - a_{n-1} + a_{n-2} - ... + (-1)^{n-1} a_1 + (-1)^n a_0$. If this alternating sum is divisible by 11, then so is the original number. For example, to test whether 2573 is divisible by 11, we compute $2 - 5 + 7 - 3 = 1$ which is not divisible by 11. We conclude that 2573 is not divisible by 11 either.

Using ideas similar to those for the divisibility rules for 3 and 9, can you explain why this rule works? Is the alternating sum always equal to the original number mod 11?

2. The divisibility rule for 7 says: to determine whether a number $a_n a_{n-1} ... a_2 a_1 a_0$ is divisible by 7, truncate (chop off) the ones digit, and subtract twice that digit from what is left after truncation (note that when we truncate, we shift all the place values down one, so we're really computing $a_n \cdot 10^{n-1} + a_{n-1} \cdot 10^{n-2} + \ldots + a_3 \cdot 10^2 + a_2 \cdot 10 + a_1 - 2 \cdot a_0$). If the resulting number is divisible by 7, then so is the original number. For example, to test whether 2573 is divisible by 7, we compute $257 - 2 \cdot 3 = 251$. This number is still quite large, so we might choose to repeat the process. We compute $25 - 2 \cdot 1 = 23$, which is not divisible by 7. We conclude that 251 is not divisible by 7, and neither is 2573.

Is the number we compute via this truncation-subtraction process always equal to the original number mod 7? What is $-2 \cdot 10$ mod 7? If we write our original number as $m = 10m' + d$, where d is the ones digit, do you see why $10m' + d$ and $m' - 2d$ might either both be divisible by 7 (that is, equal to 0 mod 7) or both not?