MATH 6 – PRACTICE MIDTERM 2

Name: SOLUTIONS.

FOR FULL CREDIT

SHOW ALL WORK

NO CALCULATORS

1. Find the area under the curve and above the x-axis from $x = 0$ to $x = \ln 2$ of the function $f(x) = \frac{e^x}{(e^x+2)^2}$.

$$\text{Area} = \int_0^{\ln 2} \frac{e^x}{(e^x+2)^2}\, dx$$

Use substitution: $u = e^x + 2$

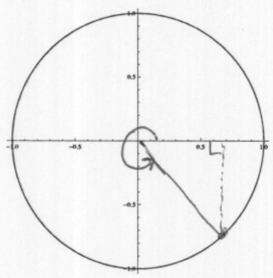
$du = e^x\, dx$.

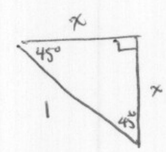
$$\text{So Area} = \int_{x=0}^{x=\ln 2} \frac{1}{u^2}\, du = -u^{-1}\Big|_{x=0}^{x=\ln 2}$$

$$= -\frac{1}{e^x+2}\Big|_{x=0}^{x=\ln 2} = -\frac{1}{e^{\ln 2}+2} - \left(-\frac{1}{e^0+2}\right)$$

$$= -\frac{1}{4} + \frac{1}{3} = \boxed{\frac{1}{12}}$$

2. Indicate on the unit circle below where the angle $\frac{7\pi}{4}$ is. Then, evaluate $\sin\frac{7\pi}{4}$, $\cos\frac{7\pi}{4}$ and $\tan\frac{7\pi}{4}$.





$x^2 + x^2 = 1$

$2x^2 = 1$

$x = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$.

$\sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$

$\cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2}$

$\tan\frac{7\pi}{4} = -1$.

3. Newton's law of cooling says that the change in temperature of an object is proportional to the difference between the object's temperature and the (constant) ambient temperature.

(a) Write a differential equation modeling Newton's law of cooling.

Let T be the function denoting temperature of our object at time t.
and T_0 be the (constant) ambient temperature.

$$\boxed{\frac{dT}{dt} = k(T - T_0)}$$

(b) Solve the differential equation you wrote in part (a).

$$\frac{1}{T-T_0}\, dT = k\, dt. \quad \text{(separate variables).}$$

Integrating both sides, we get

$$\int \frac{1}{T-T_0}\, dT = \int k\, dt = kt + C_1.$$

Use substitution: $u = T - T_0$. Then $du = dT$.

$$kt + C_1 = \int \frac{1}{u}\, du = \ln|u| = \ln|T - T_0|$$

So $|T - T_0| = e^{kt+C_1}$ and $T = Ce^{kt} + T_0$.

(c) Suppose the room is 60°F, and a cup of tea cools from 200° to 130° in 10 minutes. Write a function giving the temperature of the tea. You may leave your answer in terms of ln and powers of *e*.

Room is $60^\circ F \Rightarrow T_0 = 60$.

$$200 = Ce^{k\cdot 0} + 60 = C + 60$$

So $C = 140$

$$130 = Ce^{k\cdot 10} + 60 = 140e^{10k} + 60$$

$$70 = 140e^{10k}, \quad \text{so } \frac{1}{2} = e^{10k}.$$

$\ln(\frac{1}{2}) = 10k$ and $k = \frac{1}{10}\ln(\frac{1}{2})$

So $\boxed{T = 140e^{\frac{t}{10}\ln(\frac{1}{2})} + 60}$

(This can be rewritten as $T = 140\left(\frac{1}{2}\right)^{t/10} + 60$)

4. Suppose a bicycle wheel with 2 foot diameter makes 150 rotations per minute.

(a) Find the angular velocity of the wheel.

$$\frac{150 \text{ rotations}}{\text{minute}} \times \frac{2\pi \text{ radians}}{\text{rotation}} = \boxed{300\pi \text{ radians}/\text{minute}}$$

(b) Find the linear velocity of a point on the outside edge of the wheel.

Diameter $= 2 \Rightarrow$ radius $= 1$.

$$\text{Linear velocity} = \frac{\text{change in position}}{\text{time}} = \frac{\text{change in arc length}}{\text{time}}$$

$$= \frac{r\theta}{t}$$

$$= 1 \cdot \text{ft } 300\pi \frac{\text{radians}}{\text{minute}} = \boxed{300\pi \text{ ft}/\text{minute}}$$

5. Suppose $\sin\theta = \frac{15}{17}$ and θ is an acute angle (that is, it lies in the first quadrant). Compute $\cos\theta$, $\tan\theta$, $\sec\theta$, $\cot\theta$ and $\csc\theta$.

$\sin\theta = \frac{\text{opp}}{\text{adj}}$. θ acute $\Rightarrow$ all trig functions are positive.

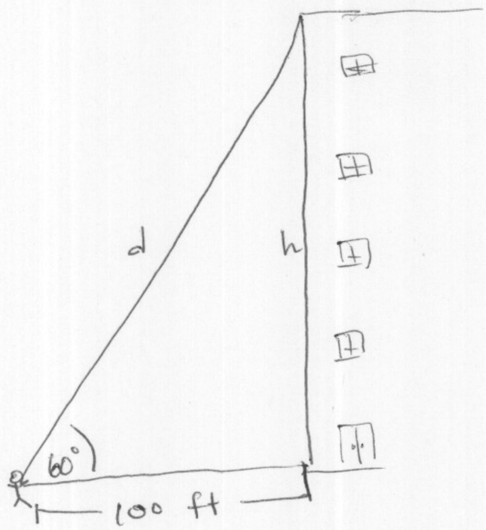
$$17^2 = 15^2 + x^2$$
$$289 = 225 + x^2$$
$$64 = x^2 \rightarrow x = 8.$$

$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{8}{17}$ $\qquad$ $\sec\theta = \frac{1}{\cos\theta} = \frac{17}{8}$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{15}{8}$ $\qquad$ $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{8/17}{15/17} = \frac{8}{15}$

$\csc\theta = \frac{1}{\sin\theta} = \frac{17}{15}$

6. You stand 100 feet from the base of a building and estimate that the angle to the top of the building is 60°. Based on this estimate, what is the height of the building? How far are you from the top of the building? (Partial credit will be given for a picture giving a visual representation of the problem.)



$$\frac{h}{100} = \tan 60° = \frac{\sin 60°}{\cos 60°} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

so $\boxed{h = 100\sqrt{3} \text{ ft}.}$

$$\frac{100}{d} = \cos 60 = \frac{1}{2}, \text{ so } \boxed{d = 200 \text{ ft}}$$

(That is, the building is about $100\sqrt{3}$ ft tall, and you are about 200 ft from the top of the building.)