MATH 6 – PRACTICE FINAL

Name: SOLUTIONS

FOR FULL CREDIT

SHOW ALL WORK

NO CALCULATORS

1. True or false: for any two functions f and g, $\int fg = \int f \int g$. If it is true, explain why. If it is false, give an example of two functions demonstrating that it is false.

This statement is false.

For example, let $f(x) = x$ and $g(x) = x$.

Then $\int f(x)g(x)\,dx = \int x^2\,dx = \frac{x^3}{3} + C$.

and $\left(\int f(x)\,dx\right)\left(\int g(x)\,dx\right) = \left(\int x\,dx\right)\left(\int x\,dx\right)$

$$= \left(\frac{x^2}{2} + C_1\right)\left(\frac{x^2}{2} + C_2\right)$$

$$= \frac{x^4}{4} + (C_1 + C_2)\frac{x^2}{2} + C_1C_2.$$

But $\frac{x^3}{3} + C \neq \frac{x^4}{4} + (C_1 + C_2)\frac{x^2}{2} + C_1C_2$

for any values of C, C_1 and C_2.

2. Compute the following indefinite integral.

$$\int \frac{\sin(\ln x)}{x}\,dx$$

Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$.

$$\int \sin u\,du = -\cos u + C$$

$$= -\cos(\ln x) + C.$$

3. Find the maximum and minimum of the function $f(x) = \sin x - \cos x$ on the interval $[0, 2\pi]$.

To find the max and min, find critical points.

$f'(x) = \cos x + \sin x = 0$

So $\sin x = -\cos x \implies x = \frac{3\pi}{4}$, or $x = \frac{7\pi}{4}$.

Test critical points and endpoints:

$f(0) = \sin(0) - \cos(0) = 0 - 1 = -1$

$f\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right) = +\frac{\sqrt{2}}{2} - \frac{-\sqrt{2}}{2} = +\sqrt{2}$

$f\left(\frac{7\pi}{4}\right) = \sin\left(\frac{7\pi}{4}\right) - \cos\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

$f(2\pi) = \sin(2\pi) - \cos(2\pi) = 0 - 1 = -1.$

The maximum is $\sqrt{2}$ (at $x = \frac{3\pi}{4}$)

The minimum is $-\sqrt{2}$ (at $x = \frac{7\pi}{4}$).

4. Find the area of the region bounded above by the graph of the function $y = \frac{e^x}{e^x+1}$, below by the x-axis, and by the lines $x = 0$ and $x = \ln 2$.

$$\text{Area} = \int_0^{\ln 2} \frac{e^x}{e^x+1}\,dx$$

Use substitution. Set $u = e^x + 1$

$du = e^x\,dx$

Change limits: $x = 0 \Rightarrow u = e^0 + 1 = 2$

$x = \ln 2 \Rightarrow u = e^{\ln 2} + 1 = 3.$

$$\text{Area} = \int_2^3 \frac{1}{u}\,du = \ln|u|\Big|_2^3 = \ln 3 - \ln 2 = \boxed{\ln\frac{3}{2}}$$

5. Compute

(a) $\sin(\arcsin \frac{2}{7})$

(b) $\arcsin(\sin \frac{5\pi}{4})$

a) $\sin\left(\arcsin \frac{2}{7}\right) = \frac{2}{7}$.

b) $\arcsin\left(\sin \frac{5\pi}{4}\right) = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

(since $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$).

6. Compute $\sin 15°$, $\cos 15°$, and $\tan 15°$.

Since 15° is in the first quadrant, all are positive.

$\sin 15° = \sin \frac{30°}{2} = \sqrt{\frac{1-\cos 30°}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \boxed{\frac{\sqrt{2-\sqrt{3}}}{2}}$

$\cos 15° = \cos \frac{30°}{2} = \sqrt{\frac{1+\cos 30°}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}$

$\tan 15° = \frac{\sin 15°}{\cos 15°} = \boxed{\frac{\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}}}$

7. Suppose a buoy moves according to simple harmonic motion as waves pass it. In other words, suppose the position of the buoy is given by $y = A\cos Bt$. The buoy moves (vertically) 3 feet from the highest point to the lowest point. It returns to the highest point every 5 seconds.

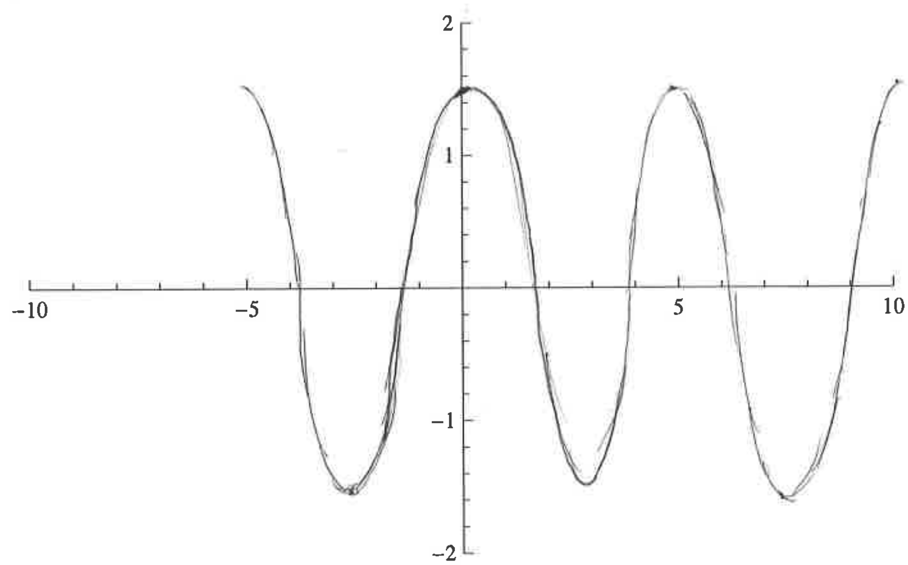
(a) Identify the amplitude and period for the function describing the position of the buoy, and use them to write the function describing the position of the buoy.

Amplitude: $\frac{3}{2} = 1.5$ ft (amplitude is half the distance from highest to lowest points).

Period: 5 seconds. $= \frac{2\pi}{B} \Rightarrow B = \frac{2\pi}{5}$

$y = \frac{3}{2}\cos\frac{2\pi t}{5}$

(b) Graph the position of the buoy on the axes below.



(c) What is the function that describes the velocity of the buoy?

Velocity is the derivative of position.

$v(t) = y' = \frac{3}{2}\cdot\left(-\sin\frac{2\pi}{5}t\right)\cdot\left(\frac{2\pi}{5}\right)$

$= \boxed{-\frac{3\pi}{5}\sin\frac{2\pi t}{5}}$

8. Suppose the rate of change in a population is proportional to the product of the current population and $\sin 12t$ where t is measured in months.

(a) Write a differential equation to model this scenario.

Let $P(t)$ = population at time t.

$$\frac{dP}{dt} = kP \cdot \sin 12t .$$

(b) Solve the differential equation you wrote in part (a).

Separate variables:

$$\frac{dP}{P} = k \sin 12t \, dt$$

Integrate

$$\int \frac{dP}{P} = k \int \sin 12t \, dt$$

$u = 12t$

$du = 12 \, dt$

$$\ln|P| = k \int \frac{1}{12} \sin u \, du$$

$$= \frac{-k}{12} \cos u + C$$

$$= \frac{-k}{12} \cos 12t + C$$

So $\boxed{P = Ce^{\frac{-k}{12} \cos 12t}}$

9. Compute the following definite integral.

$$\int_0^2 x^2\sqrt{x^3+8}\,dx$$

Let $u = x^3+8$

$du = 3x^2\,dx$, so $\frac{1}{3}du = x^2 dx$.

$$\int_{x=0}^{x=2} \frac{1}{3}\sqrt{u}\;du = \frac{1}{3}\frac{2}{3}u^{3/2}\Big|_{x=0}^{x=2} = \frac{2}{9}(x^3+8)^{3/2}\Big|_0^2$$

$$= \frac{2}{9}\cdot\left(16^{3/2} - 8^{3/2}\right)$$

$$= \frac{2}{9}\left(4^3 - 8\sqrt{8}\right)$$

$$= \boxed{\frac{2}{9}\left(64 - 16\sqrt{2}\right)}$$

10. Compute the following indefinite integral.

$$\int \frac{2x^3+3x}{x^2+1}\,dx$$

Start with polynomial long division.

$$\begin{array}{r} 2x \\ x^2+1\;\overline{)\,2x^3+3x} \\ \underline{2x^3+2x} \\ x \end{array}$$

So $\dfrac{2x^3+3x}{x^2+1} = 2x + \dfrac{x}{x^2+1}$.

$$\int \frac{2x^3+3x}{x^2+1}\,dx = \int 2x + \frac{x}{x^2+1}\,dx$$

$$= x^2 + \int \frac{x}{x^2+1}\,dx$$

Set $u = x^2+1$. Then $du = 2x\,dx$ and $\frac{1}{2}du = x\,dx$.

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int\frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C.$$

So the original integral is equal to $\boxed{x^2 + \frac{1}{2}\ln(x^2+1) + C}$

11. Find the derivative of the function

$$f(\theta) = \sec\theta \sin(\ln\theta).$$

$f'(\theta) = \sec\theta \tan\theta \sin(\ln\theta) + \sec\theta \cdot \cos(\ln\theta) \cdot \frac{1}{\theta}$

$= \boxed{\sec\theta \tan\theta \sin(\ln\theta) + \frac{\sec\theta \cos(\ln\theta)}{\theta}}$

12. Using a trig identity, compute the integral

$$\int \sin^3 x \, dx.$$

Write $\sin^2 x$ as $1-\cos^2 x$ to get

$\int (1-\cos^2 x)\sin x \, dx = \int \sin x \, dx - \int \cos^2 x \sin x \, dx$

$= -\cos x - \int \cos^2 x \sin x \, dx.$

Set $u = \cos x$.

Then $du = -\sin x \, dx$ and $-du = \sin x \, dx$.

Then we get $-\cos x + \int u^2 \, du = -\cos x + \frac{u^3}{3} + C$

$= \boxed{-\cos x + \frac{\cos^3 x}{3} + C}$

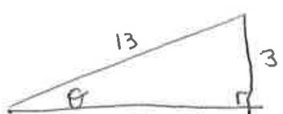
13. Verify that $\tan(\pi/2 - x) = \cot x$.

$$\tan\left(\frac{\pi}{2}-x\right) = \frac{\sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)} = \frac{\sin\frac{\pi}{2}\cos x - \sin x\cos\frac{\pi}{2}}{\cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x}.$$

But $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$, so we get

$$\tan\left(\frac{\pi}{2}-x\right) = \frac{\cos x}{\sin x} = \cot x.$$

14. Find $\sin 2\theta$ and $\cos 2\theta$ if $\sin\theta = \frac{-3}{13}$ and θ lies in the third quadrant.



$\sqrt{169-9} = \sqrt{160}$

$= 4\sqrt{10}$.

If θ is in the third quadrant, cos is negative,

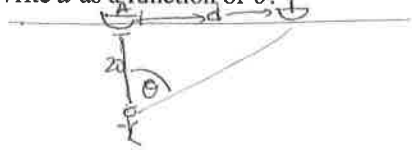
so $\cos\theta = -\frac{4\sqrt{10}}{13}$.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\left(\frac{-3}{13}\right)\left(-\frac{4\sqrt{10}}{13}\right) = \boxed{\frac{24\sqrt{10}}{169}}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{-4\sqrt{10}}{13}\right)^2 - \left(\frac{-3}{13}\right)^2 = \frac{160}{169} - \frac{9}{169} = \boxed{\frac{151}{169}}$$

15. Suppose your friend is in a boat sailing down a straight river and you are on shore watching. She starts directly across from you and 20 feet away. Let the angle between where she started and where she is now (with you, the observer as the vertex) be θ, and the distance your friend has gone be d.

(a) Write d as a function of θ.



$$\frac{d}{20} = \tan\theta, \quad \text{so} \quad \boxed{d = 20\tan\theta.}$$

(b) When θ is 60°, what is d?

$$d = 20 \cdot \tan 60^\circ = \boxed{20\sqrt{3}}$$

(c) Write your friend's speed in terms of a function of θ.

$$\boxed{d' = 20\sec^2\theta.}$$