# MATH 6 – MIDTERM 1

**Name:** SOLUTIONS.

FOR FULL CREDIT

SHOW ALL WORK

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NO CALCULATORS

1. Find the area of the region bounded by the graph of $y = x^2\sqrt{x^3+1}$ and the lines $y = 0$, $x = 0$ and $x = 2$. (10 points)

$$\text{Area} = \int_0^2 x^2\sqrt{x^3+1}\, dx.$$

Use substitution.

$u = x^3+1$

$du = 3x^2\,dx$, so $\frac{1}{3}du = x^2 dx$.

$$\int_{x=0}^{x=2} \sqrt{u}\, du = \frac{2}{3}u^{3/2}\Big|_{x=0}^{x=2} = \frac{2}{3}(x^3+1)^{3/2}\Big|_0^2$$

$$= \frac{2}{3}(9^{3/2} - 1)$$

$$= \frac{2}{3}(27-1) = \boxed{\frac{52}{3}}$$

2. Compute the indefinite integral.

$$\int \frac{x^2+x}{\sqrt{x}}\, dx$$

(10 points)

$$\int \frac{x^2+x}{\sqrt{x}}\, dx = \int \frac{x^2}{\sqrt{x}} + \frac{x}{\sqrt{x}}\, dx$$

$$= \int x^{3/2} + x^{1/2}\, dx$$

$$= \boxed{\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + C}$$

3. Compute the definite integral.

$$\int_0^1 xe^{x^2}\,dx$$

(10 points)

Use substitution. Let $u = x^2$

$du = 2x\,dx$, so $\frac{1}{2}du = x\,dx$.

Change limits: if $x = 0$, $u = 0^2 = 0$

if $x = 1$, $u = 1^2 = 1$

$$\frac{1}{2}\int_0^1 e^u\,du = \frac{1}{2}e^u\Big|_0^1 = \frac{1}{2}e - \frac{1}{2}e^0$$

$$= \boxed{\frac{e-1}{2}}$$

4. Find the solution to the following differential equation which passes through the point $(0,1)$.

$$\frac{dy}{dx} = e^{-x} + 2x + \frac{1}{\sqrt[3]{x}}$$

(10 points)

$$y = \int e^{-x} + 2x + \frac{1}{\sqrt[3]{x}}\,dx$$

$$= -e^{-x} + \frac{2x^2}{2} + \int x^{-1/3}\,dx$$

$$= -e^{-x} + x^2 + \frac{3}{2}x^{2/3} + C$$

Since $y(0) = 1$,

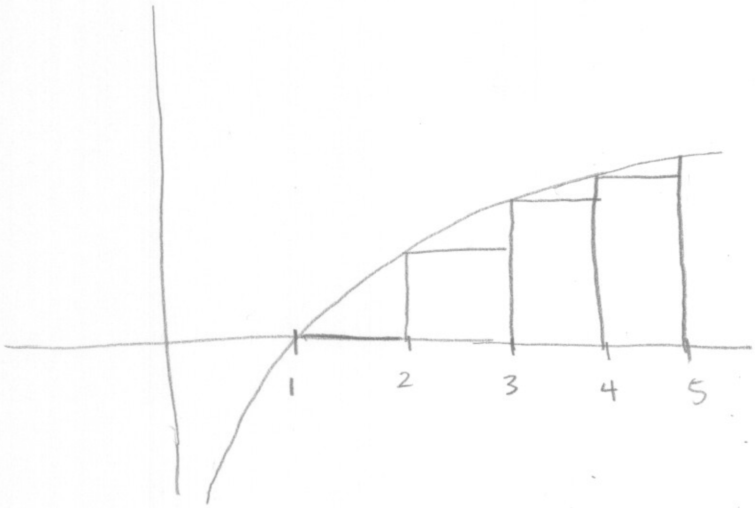
$$1 = -e^0 + 0^2 + \frac{3}{2}\,0^{2/3} + C$$

$$= -1 + C$$

So $C = 2$.

Thus, $\boxed{y = -e^{-x} + x^2 + \frac{3}{2}x^{2/3} + 2}$

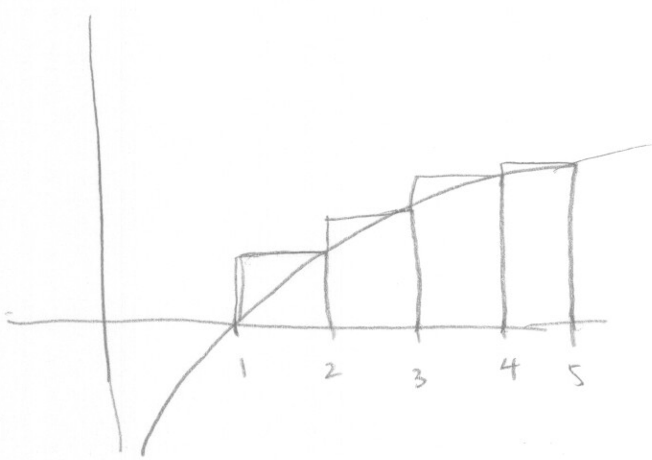
5. Estimate the area under the curve $y = \ln x$ from $x = 1$ to $x = 5$ using **four** rectangles. You may use either left or right endpoints. Partial credit will be given for a sketch of the graph and rectangles. (10 points)




With left endpoints:

$$\text{Area} \approx 1\cdot\ln 1 + 1\cdot\ln 2 + 1\cdot\ln 3 + 1\cdot\ln 4$$
$$= 0 + \ln 2 + \ln 3 + \ln 4$$
$$= \ln(2\cdot 3\cdot 4) = \boxed{\ln 24}$$

Or with right endpoints:




$$\text{Area} \approx 1\cdot\ln 2 + 1\cdot\ln 3 + 1\cdot\ln 4 + 1\cdot\ln 5$$
$$= \ln 2 + \ln 3 + \ln 4 + \ln 5$$
$$= \ln(2\cdot 3\cdot 4\cdot 5) = \boxed{\ln 120}$$

6. Let $F(x) = \int_{\frac{1}{2}}^{x} \ln t \, dt$.

(a) What is $F'(x)$? (5 points)

$\boxed{F'(x) = \ln x}$ by the second Fundamental Theorem of Calculus.

(b) Where could $F(x)$ have maxima, minima, and points of inflection? (5 points)

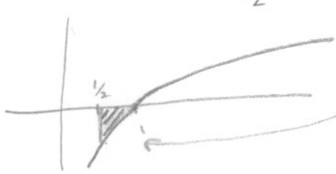
$F(x)$ has maxima, minima and points of inflection when $F'(x) = 0$

$F'(x) = \ln x = 0$ when $x = e^0 = 1$

So $F(x)$ has a maximum, minimum or point of inflection when $\boxed{x=1}$

(c) Is $F(1)$ positive, negative, or zero? (5 points)

$F(1) = \int_{1/2}^{1} \ln t \, dt$, and $\ln t$ is negative for $\frac{1}{2} \leq t \leq 1$.




F(1) measures the shaded area, and since $\ln t$ is negative in this range, the area is below the x-axis. Therefore, F(1) is $\boxed{\text{negative}}$

(d) Which is larger, $F(1)$ or $F(2)$? (5 points)

$\boxed{F(2) > F(1)}$ because $F(2) = \int_{1/2}^{2} \ln t \, dt = \int_{1/2}^{1} \ln t \, dt + \int_{1}^{2} \ln t \, dt$

$= F(1) + \int_{1}^{2} \ln t \, dt$

and $\int_{1}^{2} \ln t \, dt$ is positive (because $\ln t$ is positive from $t=1$ to $t=2$)

(e) *Extra Credit:* Classify the points you found in part (b) as maxima, minima, or points of inflection. For full credit, explain your reasoning. (5 points)

Use the second derivative test.

$F''(x) = \frac{d}{dx}(\ln x) = \frac{1}{x}$.

At $x=1$, $F''(1) = \frac{1}{1} = 1 > 0$, so $F(x)$ has a $\boxed{\text{minimum}}$ at $x=1$

Alternate solution: For $\frac{1}{2} < x < 1$, $F(x)$ is definitely negative since $\ln t$ is negative. As $x$ increases from $\frac{1}{2}$ to 1, we're getting more and more negative area. Once $x$ increases past 1, we start adding

6 e) cont.

positive area. Thus $F(x)$ decreases until $x$ reaches 1, then starts increasing. This means $F(x)$ has a $\boxed{\text{minimum}}$ at $x=1$.