Homework 3 Solutions

Section 6.5.

17. $\int 5x \sqrt[3]{1-x^2}\, dx$.

Set $u = 1-x^2$. Then $\boxed{du = -2x\,dx}$, so $-\frac{1}{2}\,du = x\,dx$.

Substitute to get:

$$\int -\frac{5}{2}\sqrt[3]{u}\,du = -\frac{5}{2}\int u^{1/3}\,du = \frac{-\frac{5}{2}u^{4/3}}{4/3} + C$$

$$= \boxed{\frac{-15}{8}(1-x^2)^{4/3} + C}$$

35. $\frac{dy}{dx} = 4x + \frac{4x}{\sqrt{16-x^2}}$

$$y = \int 4x + \frac{4x}{\sqrt{1-x^2}}\,dx = \int 4x\,dx + \int \frac{4x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{4x^2}{2} + \int \frac{4x}{\sqrt{1-x^2}}\,dx$$

Use substitution: $u = 1-x^2$

$du = -2x\,dx$, so $-2du = 4x\,dx$.

So $y = 2x^2 + \int \frac{-2}{\sqrt{u}}\,du = 2x^2 + \int -2u^{-1/2}\,du$

$$= 2x^2 + \frac{-2u^{1/2}}{1/2} + C = \boxed{2x^2 - 4\sqrt{1-x^2} + C}$$

47. $\int \frac{x^2-1}{\sqrt{2x-1}}\,dx$

Set $u = 2x-1$, so $\frac{u+1}{2} = x$ and $\left(\frac{u+1}{2}\right)^2 - 1 = \frac{u^2+2u+1}{4} - \frac{4}{4}$

Then $du = 2\,dx$, so $\frac{1}{2}du = dx$ $\quad = \frac{u^2+2u-3}{4} = x^2-1$.

Substitute to get:

$$\int \frac{1}{4}\cdot\frac{1}{2}\,\frac{u^2+2u-3}{\sqrt{u}}\,du = \frac{1}{8}\int u^{3/2} + 2u^{1/2} - 3u^{-1/2}\,du$$

$$= \frac{1}{8}\left[\frac{u^{5/2}}{5/2} + \frac{2u^{3/2}}{3/2} - \frac{3u^{1/2}}{1/2}\right] + C.$$

$$= \boxed{\frac{1}{20}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} - \frac{3}{4}(2x-1)^{1/2} + C}$$

51. $\int_{-1}^{1} x(x^2+1)^3 \, dx = ?$

Set $u = x^2+1$. Then $du = 2x\,dx$ and $\frac{1}{2}\,du = x\,dx$

Substitute to get:

$$\int_{x=-1}^{x=1} \frac{1}{2} u^3 \, du = \frac{u^4}{8}\Big|_{x=-1}^{x=1} = \frac{(x^2+1)^4}{8}\Big|_{x=-1}^{x=1}$$

$$= 2 - 2 = \boxed{0}$$ (Method 1 from class)

53. $\int_1^2 2x^2\sqrt{x^3+1} \, dx$

Set $u = x^3+1$. Then $du = 3x^2\,dx$, so $\frac{2}{3}\,du = 2x^2\,dx$.

Limits: If $x=1$, $u=2$ and if $x=2$, $u=9$.

Substitute: $$\int_2^9 \frac{2}{3}\sqrt{u}\,du = \frac{\frac{2}{3}u^{3/2}}{3/2}\Big|_2^9 = \frac{4}{9}\left(9^{3/2} - 2^{3/2}\right)$$

$$= \frac{4}{9}\left(27 - 2\sqrt{2}\right) = \boxed{12 - \frac{8\sqrt{2}}{9}}$$ (Method 2 from class)

61. $\int_5^{14} x\sqrt{x-5} \, dx$

Set $u = x-5$. Then $du = dx$, and $x = u+5$.

To change limits: If $x=5$, $u=0$
If $x=14$, $u=9$

Substitute: $$\int_0^9 (u+5)\sqrt{u}\,du = \int_0^9 u^{3/2} + 5u^{1/2}\,du$$

$$= \left(\frac{u^{5/2}}{5/2} + \frac{5u^{3/2}}{3/2}\right)\Big|_0^9$$

$$= \left(\frac{2}{5}\cdot 243 + \frac{10}{3}\cdot 27\right) - (0+0)$$

$$= \frac{486}{5} + \frac{270}{3} = \frac{1458+1350}{15} = \frac{2808}{15}$$

$$= \boxed{\frac{936}{5}}$$

14. $\int x(5x^2+4)^3\,dx$

Set $u = 5x^2+4$, so $du = 10x\,dx$

Then $\frac{1}{10}\,du = x\,dx$.

Substitute: $\frac{1}{10}\int u^3\,du = \frac{1}{10}\frac{u^4}{4} + C = \boxed{\frac{(5x^2+4)^4}{40} + C}$

16. $\int t^3\sqrt{t^4+5}\,dt$

Set $u = t^4+5$, so $du = 4t^3\,dt$.

Then $\frac{1}{4}\,du = t^3\,dt$.

Substitute: $\frac{1}{4}\int\sqrt{u}\,du = \frac{1}{4}\frac{u^{3/2}}{3/2} + C$

$= \boxed{\frac{1}{6}(t^4+5)^{3/2} + C}$

28. $\int\frac{1}{2\sqrt{x}}\,dx = \frac{1}{2}\int x^{-1/2}\,dx = \frac{\frac{1}{2}x^{1/2}}{1/2} + C = \boxed{\sqrt{x} + C}$

38. $\frac{dy}{dx} = \frac{x-4}{\sqrt{x^2-8x+1}}$

$y = \int\frac{x-4}{\sqrt{x^2-8x+1}}\,dx$

Set $u = x^2-8x+1$, so $du = 2x-8\,dx$

Then $\frac{1}{2}\,du = x-4\,dx$.

Substitute to get.

$y = \int\frac{1}{2\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du = \frac{\frac{1}{2}u^{1/2}}{1/2} + C = \boxed{\sqrt{x^2-8x+1} + C}$

46. $\int(x+1)\sqrt{2-x}\,dx =$

Set $u = 2-x$. Then $x = 2-u$ and $du = -dx$, so $-du = dx$.

Substitute: $-\int(2-u+1)\sqrt{u}\,du = -\int 3\sqrt{u} - u^{3/2}\,du$

$= -\left(3\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right) + C = \boxed{-2(2-x)^{3/2} + \frac{2}{5}(2-x)^{5/2} + C}$

54. $\int_0^1 x\sqrt{1-x^2}\,dx$.

Set $u = 1-x^2$. Then $du = -2x\,dx$ and $-\frac{1}{2}du = x\,dx$

Substitute to get

$$-\frac{1}{2}\int_{x=0}^{x=1}\sqrt{u}\,du = \left(-\frac{1}{2}\frac{u^{3/2}}{3/2}\right)\Bigg|_{x=0}^{x=1} = \left(-\frac{1}{3}(1-x^2)^{3/2}\right)\Bigg|_{x=0}^{x=1}$$

$$= 0 - \left(-\frac{1}{3}\right) = \boxed{\frac{1}{3}}$$

58. $\int_0^2 x\sqrt[3]{4+x^2}\,dx$

Set $u = 4+x^2$. Then $du = 2x\,dx$ and $\frac{1}{2}du = x\,dx$

Limits: If $x=0$, $u=4$

If $x=2$, $u=8$.

Substitute: $\int_4^8 \frac{1}{2}\sqrt[3]{u}\,du = \frac{1}{2}\frac{u^{4/3}}{4/3}\Bigg|_4^8 = \frac{3}{8}\left(8^{4/3} - 4^{4/3}\right)$

$$= \boxed{6 - \frac{3}{2}\sqrt[3]{4}}$$

62. $\int_0^1 \frac{1}{\sqrt{x+1}}\,dx$.

Set $u = x+1$ Then $du = dx$

Limits: If $x=0$, $u=1$.

If $x=1$, $u=2$.

Substitute:

$$\int_1^2 \frac{1}{\sqrt{u}}\,du = 2u^{1/2}\Big|_1^2 = \boxed{2\sqrt{2} - 2}$$

66. $\frac{dy}{dx} = 4x + \frac{9x^2}{(3x^3+1)^{3/2}}$, passing through $(0,2)$.

$$y = \int 4x + \frac{9x^2}{(3x^3+1)^{3/2}}\,dx = \int 4x\,dx + \int \frac{9x^2}{(3x^3+1)^{3/2}}\,dx$$

$$= \frac{4x^2}{2} + \int \frac{9x^2}{(3x^3+1)^{3/2}}\,dx.$$ Set $u = 3x^3+1$, then $du = 9x^2dx$.

$$= 2x^2 + \int \frac{1}{u^{3/2}}\,du = 2x^2 + \frac{u^{-1/2}}{-1/2} + C = 2x^2 - 2(3x^3+1)^{-1/2} + C.$$

Since $y(0)=2$: $2 = y(0) = 2\cdot 0 - 2(1)^{-1/2} + C = -2 + C \longrightarrow C = 4$

So $\boxed{y = 2x^2 - \frac{2}{\sqrt{3x^3+1}} + 4}$

Section 8.1

47. $\int e^{5x} (5)\,dx$.

Let $5x = u$. Then $du = 5\,dx$, so

Substitute: $\int e^u\,du = e^u + C = \boxed{e^{5x} + C}$

57. $\int \frac{5 - e^x}{e^{2x}}\,dx = \int \frac{5}{e^{2x}}\,dx - \int \frac{1}{e^x}\,dx$

$= \int 5e^{-2x}\,dx - \int e^{-x}\,dx$

For the first integral, set $u = -2x$. Then $du = -2\,dx$ and $-\frac{1}{2}du = dx$.

$\int 5e^{-2x}\,dx = -\frac{5}{2}\int e^u\,du = -\frac{5}{2}e^u + C = -\frac{5}{2}e^{-2x} + C$

For the second integral, set $u = -x$. Then $du = -dx$ and $-du = dx$.

$\int e^{-x}\,dx = -\int e^u\,du = -e^u + C = -e^{-x} + C$

Combining, we get

$$\int \frac{5 - e^x}{e^{2x}}\,dx = \boxed{-\frac{5}{2}e^{-2x} + e^{-x} + C}$$

63. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$

Let $u = 3/x$, so $du = \frac{-3}{x^2}\,dx$. Then $-\frac{1}{3}du = \frac{1}{x^2}\,dx$

If $x = 1$, $u = 3$

If $x = 3$, $u = 1$.

Substitute:

$$\int_3^1 -\frac{1}{3}e^u\,du = \left(-\frac{1}{3}e^u\right)\Big|_3^1 = -\frac{1}{3}e + \frac{1}{3}e^3 = \boxed{\frac{e^3 - e}{3}}$$

83. a) Position at time t is $x(t) = Ae^{kt} + Be^{-kt}$.

The particle is closest to the origin when $x(t)$ is minimized.

Find minima by using derivative and setting it equal to 0.

$x'(t) = Ake^{kt} - Bke^{-kt} = 0$

So $Ake^{kt} = Bke^{-kt} \Rightarrow e^{2kt} = \frac{B}{A}$, so $2kt = \ln\left(\frac{B}{A}\right)$

$x'(t) = 0$ if $t = \frac{1}{2k}\ln\left(\frac{B}{A}\right)$

83 a) cont.

Check to see that this is a minimum (and not a maximum or point of inflection) using the second derivative:

$$x''(t) = Ak^2 e^{kt} + Bk^2 e^{-kt}$$

$$x''\left(\frac{1}{2k}\ln\left(\frac{B}{A}\right)\right) = Ak^2 e^{\frac{1}{2}\ln\left(\frac{B}{A}\right)} + Bk^2 e^{-\frac{1}{2}\ln\left(\frac{B}{A}\right)}$$

$$= Ak^2\sqrt{\frac{B}{A}} + Bk^2\sqrt{\frac{A}{B}} = \sqrt{AB}\,k^2 + \sqrt{AB}\,k^2$$

$$= 2\sqrt{AB}\,k^2 > 0$$

So there is in fact a minimum at

$$\boxed{t = \frac{1}{2k}\ln\left(\frac{B}{A}\right).}$$

b) Acceleration is given by $x''(t) = Ak^2e^{kt} + Bk^2e^{-kt}$

$$= k^2\left(Ae^{kt} + Be^{-kt}\right)$$

$$= k^2 x(t)$$

The constant of proportionality is $\boxed{k^2.}$

50. $\int x^2 e^{x^3/2}\,dx$

set $u = \frac{x^3}{2}$. then $du = \frac{3}{2}x^2\,dx$ and $\frac{2}{3}du = x^2\,dx$

Substitute to get

$$\int \frac{2}{3}e^u\,du = \frac{2}{3}e^u + C = \boxed{\frac{2}{3}e^{x^3/2} + C}$$

52. $\int \frac{e^{1/x^2}}{x^3}\,dx$

set $u = \frac{1}{x^2}$. Then $du = \frac{-2}{x^3}\,dx$, so $-\frac{1}{2}du = \frac{1}{x^3}\,dx$

Substitute:

$$-\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = \boxed{-\frac{1}{2}e^{1/x^2} + C}$$

54. $\int e^{2x}(1-3e^{2x})^2\,dx$.

Let $u = 1-3e^{2x}$. Then $du = -6e^{2x}\,dx$, and $-\frac{1}{6}\,du = e^{2x}\,dx$.

Substitute to get

$$-\frac{1}{6}\int u^2\,du = -\frac{1}{6}\frac{u^3}{3} + C = \frac{-u^3}{18} + C = \boxed{\frac{-(1-3e^{2x})^3}{18} + C}$$

56. $\int e^x(e^x - e^{-x})\,dx = \int e^{2x}\,dx - \int 1\,dx$

Set $u = 2x$. Then $du = 2\,dx$ and $\frac{1}{2}\,du = dx$.

Substitute:

$$\frac{1}{2}\int e^u\,du - \int dx$$

$$= \frac{1}{2}e^u - x + C = \boxed{\frac{1}{2}e^{2x} - x + C}$$

58. $\int \frac{e^{2x}}{(1+e^{2x})^2}\,dx =$

Set $u = 1+e^{2x}$. Then $du = 2e^{2x}\,dx$ and $\frac{1}{2}\,du = e^{2x}\,dx$.

Substitute:

$$\int \frac{1}{2}\,\frac{1}{u^2}\,du = \frac{1}{2}\cdot\frac{-1}{u} + C = \boxed{\frac{-1}{2(1+e^{2x})} + C}$$

62. $\int_3^4 e^{3-x}\,dx$.

Set $u = 3-x$. Then $du = -dx$ and $-du = dx$.

Substitute:

$$-\int_{x=3}^{x=4} e^u\,du = -e^u\Big|_{x=3}^{x=4} = \left(-e^{3-x}\right)\Big|_{x=3}^{x=4} = -\frac{1}{e} - (-e^0)$$

$$= \boxed{1 - \frac{1}{e}}$$

64. $\int_0^{\sqrt{2}} x e^{-(x^2/2)} \, dx$

Set $u = -\frac{x^2}{2}$. Then $du = -x\,dx$ and $-du = x\,dx$.

Limits: If $x=0$, $u=0$. If $x=\sqrt{2}$, $u=-1$.

Substitute: $\int_0^{-1} - e^u \, du = \left(-e^u\right)\Big|_0^{-1} = -\frac{1}{e} - (-e^0) = \boxed{1-\frac{1}{e}}$

68. $\int_{-3}^{3} \frac{e^{2x}+2e^x+1}{e^x} dx = \int_{-3}^{3} e^x + 2 + e^{-x} \, dx$

$= \int_{-3}^{3} e^x \, dx + \int_{-3}^{3} 2 \, dx + \int_{-3}^{3} e^{-x} \, dx$

$= e^x\Big|_{-3}^{3} + 2x\Big|_{-3}^{3} + \int_{-3}^{3} e^{-x} \, dx$

$= (e^3 - e^{-3}) + (6-(-6)) + \int_{-3}^{3} e^{-x} \, dx.$

Set $u=-x$. Then $du = -dx$ and $-du = dx$.

$\int_{-3}^{3} e^{-x} \, dx = -\int_{x=-3}^{x=3} e^u \, du = -e^u\Big|_{x=-3}^{x=3} = \left(-e^{-x}\right)\Big|_{x=-3}^{x=3}$

$= -e^{-3} - (-e^3) = e^3 - e^{-3}$

So combining the pieces, we get.

$e^3 - e^{-3} + 12 + e^3 - e^{-3} = \boxed{2e^3 - 2e^{-3} + 12}$

74. $f''(x) = x + e^{2x}$ $f(0) = \frac{1}{4}$, $f'(0) = \frac{1}{2}$

$f'(x) = \int f''(x) \, dx = \int x + e^{2x} \, dx = \frac{x^2}{2} + \int e^{2x} \, dx$

Set $u = 2x$. Then $du = 2\,dx$ and $\frac{1}{2} du = dx$

$f'(x) = \frac{x^2}{2} + \frac{1}{2}\int e^u \, du = \frac{x^2}{2} + \frac{1}{2}e^u + C = \frac{x^2}{2} + \frac{1}{2}e^{2x} + C.$

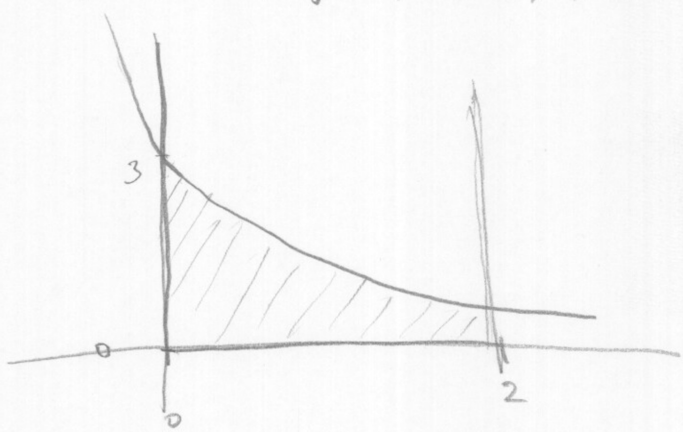
$f'(0) = \frac{1}{2} = 0 + \frac{1}{2}e^0 + C = \frac{1}{2} + C \longrightarrow C = 0$

$f(x) = \int f'(x) \, dx = \int \frac{x^2}{2} + \frac{1}{2}e^{2x} \, dx = \frac{x^3}{6} + \frac{1}{2}\int e^{2x} \, dx = \frac{x^3}{6} + \frac{1}{4}e^{2x} + C$

$f(0) = \frac{1}{4} = 0 + \frac{1}{4}e^0 + C = \frac{1}{4} + C \longrightarrow C = 0$

So $\boxed{f(x) = \frac{x^3}{6} + \frac{1}{4}e^{2x}}$

78. $y = e^{-2x} + 2,\ y = 0,\ x = 0,\ x = 2$



$$\text{Area} = \int_0^2 e^{-2x} + 2\, dx = \int_0^2 e^{-2x}\, dx + \int_0^2 2\, dx$$

Set $u = -2x$. Then $du = -2dx$, and $-\frac{1}{2} du = dx$.

Substitute:

$$\text{Area} = \int_{x=0}^{x=2} -\frac{1}{2} e^u\, du + \int_0^2 2\, dx$$

$$= -\frac{1}{2} e^u \Big|_{x=0}^{x=2} + 2x\Big|_0^2$$

$$= -\frac{1}{2} e^{-2x} \Big|_{x=0}^{x=2} + (4-0)$$

$$= -\frac{1}{2} e^{-4} - -\frac{1}{2} e^0 + 4$$

$$= 4 + \frac{1}{2} - \frac{1}{2e^4}$$

$$= \boxed{\frac{9}{2} - \frac{1}{2e^4}}$$

82. $\int_0^x 0.3 e^{-0.3t}\, dt = \frac{1}{2}$ Solve for x.

Set $u = -0.3t$. Then $du = -0.3\, dt$ and $-du = 0.3\, dt$.

Substitute: $\int_{t=0}^{t=x} -e^u\, du = -e^u \Big|_{t=0}^{t=x} = -e^{-0.3t}\Big|_{t=0}^{t=x} = -e^{-0.3x} + e^0 = -e^{-0.3x} + 1$

Solve $\frac{1}{2} = -e^{-0.3x} + 1 \implies \frac{1}{2} = e^{-0.3x}$.

$\ln \frac{1}{2} = -0.3x$, so $x = -\frac{10}{3} \ln \frac{1}{2} = \boxed{\frac{10}{3} \ln 2}$