Homework 1 Solutions

Section 6.1

1. Verify that $\int \left(-\frac{6}{x^4}\right) dx = \frac{2}{x^3} + C$

$$\frac{d}{dx}\left(\frac{2}{x^3} + C\right) = \frac{d}{dx}\left(2x^{-3} + C\right)$$

$$= -6x^{-4} = \frac{-6}{x^4} \checkmark$$

11. $\int \frac{1}{x\sqrt{x}}\, dx = \int \frac{1}{x^{3/2}}\, dx = \int x^{-3/2}\, dx$

$$= \frac{x^{-1/2}}{-1/2} + C = -2x^{-1/2} + C$$

$$= \boxed{\frac{-2}{\sqrt{x}} + C}$$

43. $\frac{dy}{dx} = 3x^2 - 1$, solution passes through $(0,2)$

$y = \int 3x^2 - 1\, dx = x^3 - x + C$

Passing through $(0,2)$ means $y(0) = 2$.

$2 = y(0) = 0^3 - 0 + C = C$.

So $\boxed{y = x^3 - x + 2}$

59. Ball thrown upward with initial velocity $60\ \text{ft/sec}$. from a height of 6 ft.

Let $y(t)$ be the position at time t, $v(t)$ be the velocity at time t and $a(t)$ be the acceleration at time t.

Given $y(0) = 6$, $v(0) = 60$, $a(t) = -32$.

Since velocity is the rate of change of position, $v(t) = y'(t)$

Since acceleration is the rate of change of velocity, $a(t) = v'(t)$.

So $v(t) = \int a(t)\, dt = \int -32\, dt = -32t + C$

Since $v(0) = 60$, $v(0) = -32(0) + C \Rightarrow C = 60$.

$v(t) = -32t + 60$.

59 continued.

Now since $y'(t) = v(t)$, $y(t) = \int v(t)\,dt = \int -32t + 60\,dt$

$$= -32 \cdot \frac{t^2}{2} + 60t + C$$

$$= -16t^2 + 60t + C$$

Since $y(0) = 6$, $C = 6$ $\quad (-16(0)^2 + 60(0) + C = 6)$.

How high will the ball go?

That's the same as asking: What is the maximum value of $y(t)$?

Find the max. value of $y(t)$ by diferentiating and setting $y'(t) = 0$:

$$y'(t) = -32t + 60 = 0 \quad \text{when } t = \frac{-60}{-32} = \frac{15}{8}.$$

The ball will go up to $y\left(\frac{15}{8}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) + 6 = \boxed{62.25 \text{ ft}}$

71. A particle moves along the x-axis with velocity $v(t) = \frac{1}{\sqrt{t}}$, $t > 0$.

At time $t = 1$, position is $x = 4$.

Acceleration is rate of change of velocity:

$$a(t) = v'(t) = \frac{d}{dt}\frac{1}{\sqrt{t}} = \frac{d}{dt} t^{-1/2} = \boxed{-\frac{1}{2} t^{-3/2}}$$

Position is obtaned by integrating velocity:

$$y(t) = \int v(t)\,dt = \int \frac{1}{\sqrt{t}}\,dt = \int t^{-1/2}\,dt = \frac{t^{1/2}}{1/2} + C$$

$$= 2t^{1/2} + C$$

$$y(1) = 4 = 2 \cdot \sqrt{1} + C \longrightarrow C = 2$$

So position is $\boxed{y(t) = 2\sqrt{t} + 2}$

14. $\int \frac{1}{(3x)^2}\,dx = \int \frac{1}{3^2x^2}\,dx = \int \frac{1}{9} \cdot \frac{1}{x^2}\,dx = \frac{1}{9}\int \frac{1}{x^2}\,dx = \frac{1}{9}\left(\int x^{-2}\,dx\right)$

$$= \frac{1}{9}\,\frac{x^{-1}}{-1} + C = \boxed{-\frac{1}{9x} + C}$$

16. $\int (13 - x)\,dx = \int 13\,dx - \int x\,dx = \boxed{13x - \frac{x^2}{2} + C}$

Check: $\frac{d}{dx}\left(13x - \frac{x^2}{2} + C\right) = 13 - \frac{2x}{2} + 0$

$$= 13 - x \checkmark$$

§6.1 cont.

18. $\int (8x^3 - 9x^2 + 4)\,dx = 8\int x^3\,dx - 9\int x^2\,dx + \int 4\,dx$

$= 8\cdot\frac{x^4}{4} - 9\cdot\frac{x^3}{3} + 4x + C$

$= \boxed{2x^4 - 3x^3 + 4x + C}$

Check: $\frac{d}{dx}(2x^4 - 3x^3 + 4x + C) = 8x^3 - 9x^2 + 4 + 0$ ✓

22. $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx = \int x^{1/2}\,dx + \frac{1}{2}\int x^{-1/2}\,dx = \frac{x^{3/2}}{3/2} + \frac{1}{2}\cdot\frac{x^{1/2}}{1/2} + C$

$= \boxed{\frac{2}{3}x^{3/2} + x^{1/2} + C}$

Check: $\frac{d}{dx}\left(\frac{2}{3}x^{3/2} + x^{1/2} + C\right) = x^{1/2} + \frac{1}{2}x^{-1/2} + 0$ ✓

24. $\int \left(\sqrt[4]{x^3} + 1\right) dx = \int x^{3/4}\,dx + \int 1\,dx = \frac{x^{7/4}}{7/4} + x + C = \boxed{\frac{4}{7}x^{7/4} + x + C}$

Check: $\frac{d}{dx}\left(\frac{4}{7}x^{7/4} + x + C\right) = x^{3/4} + 1 + 0$ ✓

28. $\int \frac{x^2 + 2x - 3}{x^4}\,dx = \int\left(\frac{1}{x^2} + \frac{2}{x^3} - \frac{3}{x^4}\right)dx = \int \frac{1}{x^2}\,dx + 2\int\frac{1}{x^3}\,dx - 3\int\frac{1}{x^4}\,dx$

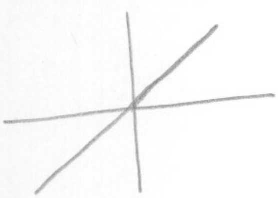
$= \frac{x^{-1}}{-1} + 2\,\frac{x^{-2}}{-2} - 3\,\frac{x^{-3}}{-3} + C = \boxed{-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C}$

Check: $\frac{d}{dx}\left(-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C\right) = \frac{1}{x^2} + 2\frac{1}{x^3} - 3\frac{1}{x^4} + 0$ ✓

32. $\int (1 + 3t)t^2\,dt = \int (t^2 + 3t^3)\,dt = \boxed{\frac{t^3}{3} + 3\frac{t^4}{4} + C}$

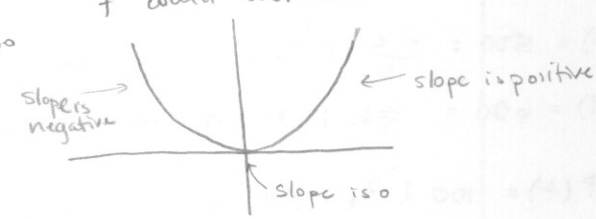
Check: $\frac{d}{dt}\left(\frac{t^3}{3} + \frac{3}{4}t^4 + C\right) = t^2 + 3t^3 + 0$ ✓

38. f' looks like

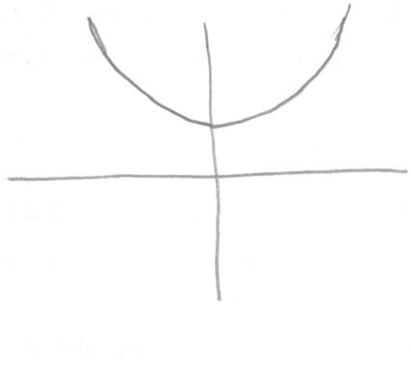


, so

f could look like



or f could look like



(same, but shifted up).

44. $\frac{dy}{dx} = -\frac{1}{x^2}$, $x > 0$ passing through $(1, 3)$.

$$y = \int -\frac{1}{x^2}\,dx = \int -x^{-2}\,dx = -\frac{x^{-1}}{-1} + C = \frac{1}{x} + C.$$

$$y(1) = 3 = \frac{1}{1} + C \rightarrow C = 2.$$

$$\boxed{y = \frac{1}{x} + 2}$$

50. $g'(x) = 6x^2$, $g(0) = -1$.

$$g(x) = \int 6x^2\,dx = 6\,\frac{x^3}{3} + C = 2x^3 + C.$$

$$g(0) = -1 = 2(0)^3 + C \Rightarrow -1 = C$$

$$\boxed{g(x) = 2x^3 - 1}$$

54. $f''(x) = x^2$, $f'(0) = 8$, $f(0) = 4$

$$f'(x) = \int f''(x)\,dx = \int x^2\,dx = \frac{x^3}{3} + C$$

$$f'(0) = 8 = \frac{0^3}{3} + C \longrightarrow C = 8$$

$$f(x) = \int f'(x)\,dx = \int \frac{x^3}{3} + 8\,dx = \frac{1}{3}\,\frac{x^4}{4} + 8x + C = \frac{1}{12}x^4 + 8x + C$$

$$f(0) = 4 = \frac{1}{12}\cdot 0^4 + 8(0) + C \longrightarrow C = 4.$$

$$\boxed{f(x) = \frac{x^4}{12} + 8x + 4}$$

56. $\frac{dP}{dt} = k\sqrt{t}$, $P(0) = 500$, $P(1) = 600$ Find $P(7)$.

$$P(t) = \int k\sqrt{t}\,dt = k\int t^{1/2}\,dt = \frac{k\,t^{3/2}}{3/2} + C$$

$$P(t) = \frac{2k}{3}t^{3/2} + C$$

$$P(0) = 500 = \frac{2k}{3}\cdot 0^{3/2} + C \rightarrow C = 500.$$

$$P(1) = 600 = \frac{2}{3}k\cdot 1 + 500 \rightarrow k = 150.$$

So $P(t) = 100\,t^{3/2} + 500$ and $P(7)$ = population after 7 days $= 100\cdot 7^{3/2} + 500$
$\approx \boxed{2352}$

1852
500
352
18.52

60. Show that the height above the ground of an object thrown upward from a point s_0 ft. above the ground with initial velocity v_0 is given by $-16t^2 + v_0t + s_0$.

Acceleration $a(t) = -32$ ft/s²

Velocity $v(t) = \int a(t)\,dt = \int -32\,dt = -32t + C.$

Since $v(0) = v_0$, $-32\cdot 0 + C = v_0 \Rightarrow v_0 = C.$

$\therefore v(t) = -32t + v_0.$

Position: $f(t) = \int v(t)\,dt = \int -32t + v_0\,dt = -32\frac{t^2}{2} + v_0t + C = -16t^2 + v_0t + C.$

Since $f(0) = s_0$, $-16\cdot 0^2 + v_0\cdot 0 + C = s_0 \longrightarrow C = s_0.$

So $\boxed{f(t) = -16t^2 + v_0t + s_0}$ ✓

§6.1 cont.

72. a) It takes 13 seconds to accelerate from 25 km/hr to 80 km/hr. Assuming constant acceleration, compute a(t) in m/s^2

$$25 \frac{km}{hr} = \frac{25\,0\cancel{00}\ m}{36\cancel{00}\ s} = \frac{250}{36}\frac{m}{s}$$

$$80 \frac{km}{hr} = \frac{80{,}0\cancel{00}\ m}{36\cancel{00}\ s} = \frac{800}{36}\frac{m}{s}$$

$$a(t) = \frac{\frac{800}{36} - \frac{250}{36}}{13}\ m/s^2 = \frac{550}{36\cdot 13}\ m/s^2 \approx 1.18\ m/s^2$$

$$v(t) = \int a(t)\,dt = \int 1.18\,dt = 1.18t + C$$

Since $v(0) = 25$, $1.18(0) + C = 25 \to C = 25$

$v(t) = 1.18t + 25$

Position $y(t) = \int v(t)\,dt = \int 1.18t + 25\,dt = 1.18\frac{t^2}{2} + 25t + C$.

Letting initial position be 0, we just need position at time 13

$y(0) = 0 = 1.18\cdot\frac{0^2}{2} + 25\cdot 0 + C \longrightarrow C = 0$.

$y(13) = 1.18\left(\frac{(13)^2}{2}\right) + 25\cdot 13 + 0 = \boxed{424.3 \text{ meters}}$

Section 6.2

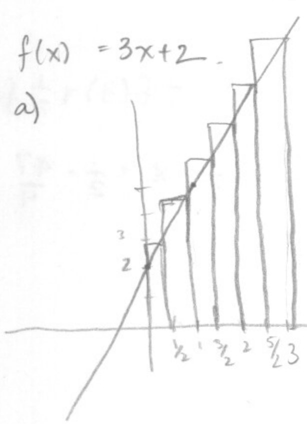
3. $\sum_{k=0}^{4} \frac{1}{k^2+1} = \frac{1}{0^2+1} + \frac{1}{1^2+1} + \frac{1}{2^2+1} + \frac{1}{3^2+1} + \frac{1}{4^2+1}$

$= 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \frac{1}{17}$

$= \boxed{\frac{931}{510} \approx 1.825}$

25. $f(x) = 3x+2$.

a)



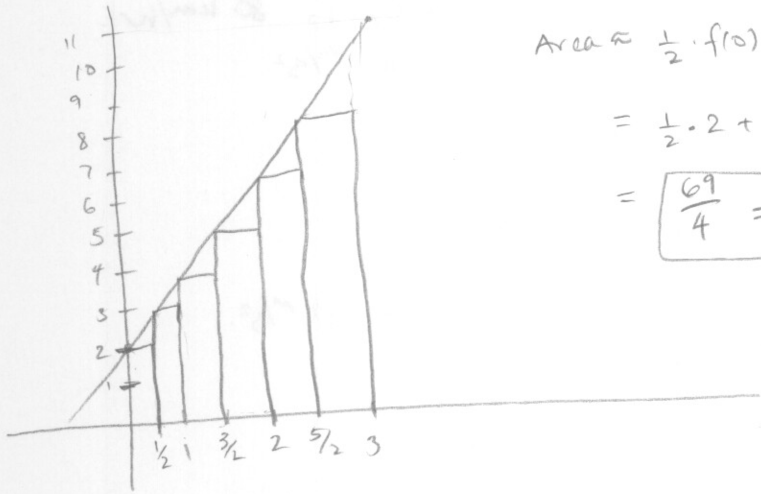
Estimate area between graph and x-axis from x=0 to x=3 using 6 rectangles and right endpoints.

Area $\approx \frac{1}{2}\cdot f(\frac{1}{2}) + \frac{1}{2}f(1) + \frac{1}{2}f(\frac{3}{2}) + \frac{1}{2}f(2) + \frac{1}{2}f(\frac{5}{2}) + \frac{1}{2}f(3)$

$= \frac{1}{2}\cdot\frac{7}{2} + \frac{1}{2}\cdot 5 + \frac{1}{2}\cdot\frac{13}{2} + \frac{1}{2}\cdot 8 + \frac{1}{2}\cdot\frac{19}{2} + \frac{1}{2}\cdot 11$

$= \boxed{\frac{87}{4} = 21.75}$

25 b) Use left endpoints



$$\text{Area} \approx \tfrac{1}{2}\cdot f(0) + \tfrac{1}{2}\cdot f(1/2) + \tfrac{1}{2}\cdot f(1) + \tfrac{1}{2} f(3/2) + \tfrac{1}{2} f(2) + \tfrac{1}{2} f(5/2)$$

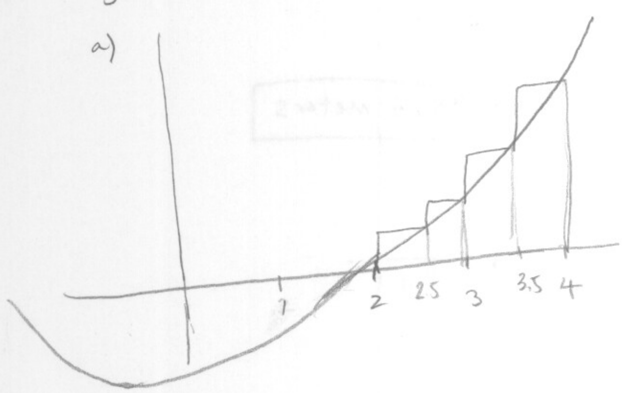
$$= \tfrac{1}{2}\cdot 2 + \tfrac{1}{2}\cdot \tfrac{7}{2} + \tfrac{1}{2}\cdot 5 + \tfrac{1}{2}\cdot \tfrac{13}{2} + \tfrac{1}{2}\cdot 8 + \tfrac{1}{2}\cdot \tfrac{19}{2}$$

$$= \boxed{\tfrac{69}{4} = 17.25}$$

4. $\displaystyle\sum_{j=4}^{7} \frac{2}{j} = \frac{2}{4} + \frac{2}{5} + \frac{2}{6} + \frac{2}{7} = \boxed{\frac{319}{210} \approx 1.519}$

26. $g(x) = x^2 + x - 4$.

a)



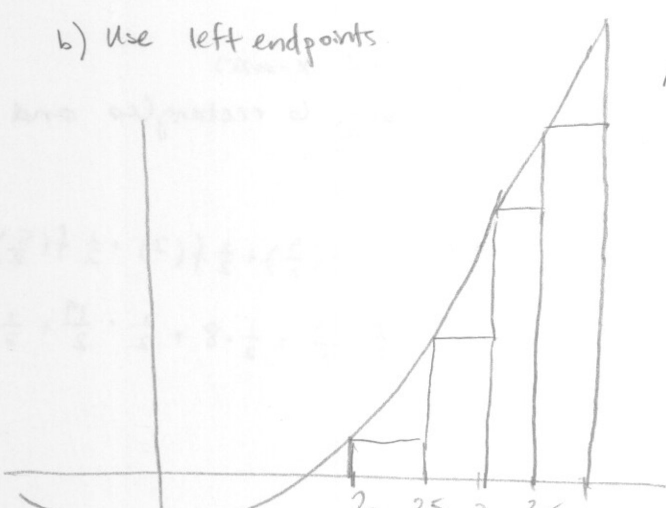
Right endpoints:

$$\text{Area} \approx \tfrac{1}{2} f(2.5) + \tfrac{1}{2} f(3) + \tfrac{1}{2} f(3.5) + \tfrac{1}{2} f(4)$$

$$= \tfrac{1}{2}\cdot\tfrac{19}{4} + \tfrac{1}{2}\cdot 8 + \tfrac{1}{2}\cdot\tfrac{47}{4} + \tfrac{1}{2}\cdot 16$$

$$= \boxed{\tfrac{81}{4} = 20.25}$$

b) Use left endpoints



$$\text{Area} \approx \tfrac{1}{2} f(2) + \tfrac{1}{2} f(2.5) + \tfrac{1}{2} f(3) + \tfrac{1}{2} f(3.5)$$

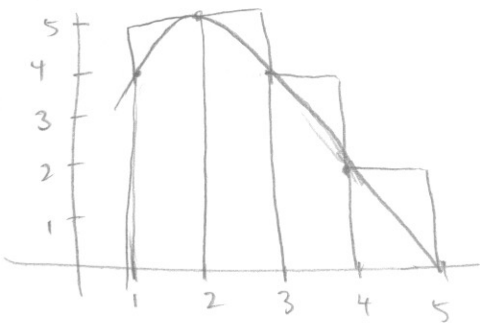
$$= \tfrac{1}{2}\cdot 2 + \tfrac{1}{2}\cdot\tfrac{19}{4} + \tfrac{1}{2}\cdot 8 + \tfrac{1}{2}\cdot\tfrac{47}{4}$$

$$= \boxed{\tfrac{53}{4} = 13.25}$$

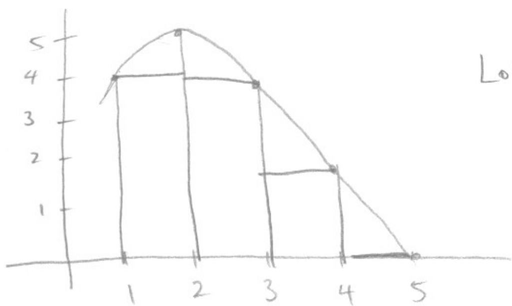
§6.2 cont.

28. Bound the area with upper and lower sums.

Upper sum:



Upper sum $= 1\cdot 5 + 1\cdot 5 + 1\cdot 4 + 1\cdot 2$
$= 16$



Lower sum: $1\cdot 4 + 1\cdot 4 + 1\cdot 2 + 1\cdot 0$
$= 10.$

$\boxed{10 \leq \text{Area} \leq 16}$

Section 6.3

13. $f(x) = 5.$

$\boxed{\int_0^4 5\,dx}$

19. $g(y) = y^3$

$\boxed{\int_0^2 y^3\,dy}$

31. $\int_4^2 x\,dx = -\int_2^4 x\,dx = \boxed{-6}$

18. $f(x) = \dfrac{4}{x^2+2}$

$\boxed{\int_{-1}^{1} \dfrac{4}{x^2+2}\,dx}$

38. $\int_2^4 (10+4x-3x^3)\,dx = 10\int_2^4 dx + 4\int_2^4 x\,dx - 3\int_2^4 x^3\,dx$

$= 10\cdot 2 + 4\cdot 6 - 3\cdot 60$

$= 20 + 24 - 180$

$= \boxed{-136}$

44. a) $\int_0^1 -f(x)\,dx = -\int_0^1 f(x)\,dx$

$= -\left(\frac{1}{2}\cdot 1\cdot 1\right) = \boxed{+\frac{1}{2}}$

b) $\int_3^4 3f(x)\,dx = 3\int_3^4 f(x)\,dx = 3\cdot 1\cdot 2 = \boxed{6}$

c) $\int_0^7 f(x)\,dx =$ Area of trapezoid $-$ Area of 2 triangles

$= \frac{5+1}{2}\cdot 2 - 2\left(\frac{1}{2}\cdot 1\cdot 1\right)$

$= 6 - 1 = \boxed{5}$

d) $\int_5^{11} f(x)\,dx =$ Area of 2 right triangles $-$ Area of triangle below x-axis

$= 2\left(\frac{1}{2}\cdot 1\cdot 1\right) - \frac{1}{2}\cdot 4\cdot 2$

$= 1 - 4 = \boxed{-3}$

triangle as in d

e) $\int_0^{11} f(x)\,dx = -\frac{1}{2} + 6 - 4 + \frac{1}{2} = \boxed{2}$

triangle as in c | trapezoid as in c | triangle from d below x-axis

f) $\int_4^{10} f(x)\,dx =$ Area of triangle above x-axis $-$ area of triangle below x-axis

$= \frac{1}{2}\cdot 2\cdot 2 - \frac{1}{2}\cdot 4\cdot 2 = 2 - 4 = \boxed{-2}$