MATH 100 – PRACTICE EXAM 3

Name: SOLUTIONS

Lecture: MWF 12 MWF 1 MWF 3

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FOR FULL CREDIT, SHOW ALL WORK
NO CALCULATORS

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2	
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1. Let $f(x) = \arctan x$.

(a) Find the location of all local minima and local maxima for f. (3 points)

$f'(x) = \frac{1}{1+x^2}$

Critical points are where f' doesn't exist or where $f' = 0$

$f'(x)$ exists for all x and is never 0, so there are $\boxed{\text{no max or min}}$

(b) Find the intervals where f is increasing and where f is decreasing. (2 points)

f increasing when $f' > 0$ and decreasing when $f' < 0$.

$\frac{1}{1+x^2} > 0$ for all x, so $\boxed{f \text{ is increasing on } (-\infty, \infty)}$

(c) Find the intervals where f is concave up and the intervals where f is concave down. (3 points)

$f''(x) = -\frac{1}{(1+x^2)^2} \cdot 2x = \frac{-2x}{(1+x^2)^2}$

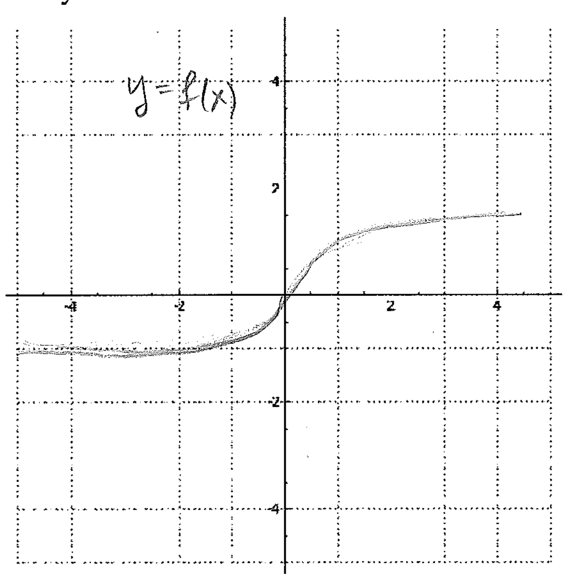
f is concave up when $f''(x) > 0$, i.e. when $x < 0$

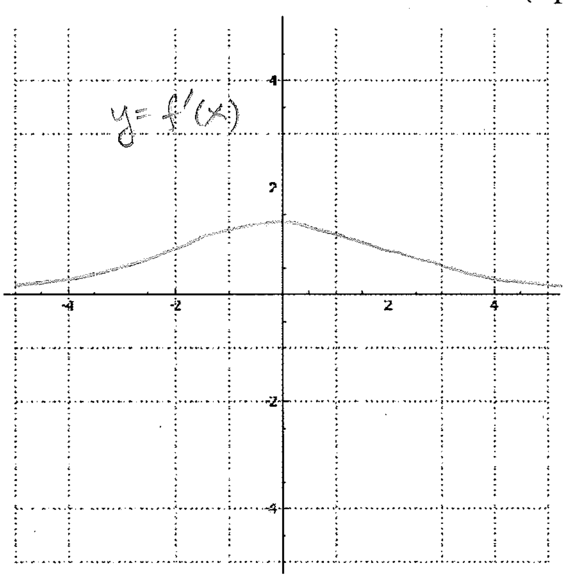
f is concave down when $f''(x) < 0$, ie when $x > 0$.

(d) Find all points of inflection for f. (2 points)

Points of inflection are points where f changes concavity, i.e. at $\boxed{x = 0}$

(e) Sketch a graph of $f(x)$ and of $f'(x)$. Indicate which is the graph of f and which is the graph of f' clearly. (5 points)





2. Find all points (x, y) where the line tangent to the curve defined by

$$x^2 - \frac{x}{y^2+1} = \frac{1}{3}$$

are horizontal. (8 points)

Use implicit differentiation.

$$2x - \frac{(y^2+1) - x\left(2y\frac{dy}{dx}\right)}{(y^2+1)^2} = 0$$

Tangent Line is horizontal when $\frac{dy}{dx} = 0$.

Plug in $\frac{dy}{dx} = 0$.

$$2x - \frac{y^2+1}{(y^2+1)^2} = 0$$

$$2x - \frac{1}{y^2+1} = 0$$

$$2x = \frac{1}{y^2+1}$$

$$x = \frac{1}{2(y^2+1)}$$

Plug back into the original equation to find points on the curve:

$$\frac{1}{4(y^2+1)^2} - \frac{1}{2(y^2+1)(y^2+1)} = \frac{1}{3}.$$

$$\left(\frac{1}{4} - \frac{1}{2}\right)\frac{1}{(y^2+1)^2} = \frac{1}{3}$$

$$\frac{1}{(y^2+1)^2} = -\frac{4}{3}$$

No real solutions exist, so the tangent line is never horizontal.

3. Compute $g'(1)$ where $g(x)$ is the inverse function of $f(x) = x^{\log_2 x}$. (Hint: Express f as a power of e or use logarithmic differentiation.) (9 points)

$$g'(1) = \frac{1}{f'(g(1))}.$$

First find $f'(x)$:

$$\ln f(x) = \ln x^{\log_2 x} = (\log_2 x)(\ln x).$$

$$(\ln f(x))' = \frac{f'(x)}{f(x)} = \frac{1}{\ln 2} \cdot \frac{1}{x} \cdot \ln x + \log_2 x \cdot \frac{1}{x}$$

$$= \frac{1}{x}\left(\frac{\ln x}{\ln 2} + \log_2 x\right)$$

Now: what is $g(1)$?

If $g(1) = a$, $f(a) = 1$, i.e. $a^{\log_2 a} = 1$.

So $a = 1$ $\quad (1^{\log_2 1} = 1^0 = 1)$.

So $g'(1) = \dfrac{1}{f'(1)} = \dfrac{1}{\frac{1}{1}\left(\frac{\ln 1}{\ln 2} + \log_2 1\right)} = \dfrac{1}{1(0+0)}$

$$= \frac{1}{0} \quad \boxed{\text{undefined}}$$

4. Use Rolle's Theorem to explain why $f(x) = x^5 + 16x$ has exactly one real root. (7 points)

Note that $x = 0$ is a root.

$f(0) = 0^5 + 16 \cdot 0 = 0 + 0 = 0$.

So we just need to explain why this is the only root.

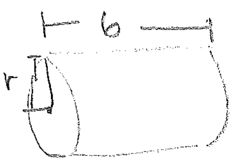
Rolle's thm says that if we had another root, say b, then in $(0, b)$, there would be a point c with $f'(c) = 0$ (since $f(0) = f(b) = 0$).

But $f'(x) = 5x^4 + 16 > 0$ for all x.

So f can't have another root b.

5. A cylindrical roll of toilet paper is 6 inches wide. Suppose you use the toilet paper at a constant rate (that is, the volume of the toilet paper roll changes at a constant rate of $-k$ cubic inches per week).

(a) How quickly is the radius of the toilet paper roll changing when the radius is 2 inches? Include units and state your answer in a complete sentence. (7 points)



$$V(t) = \pi r(t)^2 \cdot h = 6\pi r(t)^2.$$

$$\frac{dV}{dt} = 12\pi r(t) \cdot \frac{dr}{dt}.$$

$$-k = 12\pi r(t) \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-k}{12\pi r(t)}$$

When the radius is 2 inches, $\frac{dr}{dt} = \frac{-k}{24\pi}$ and the radius is changing at a rate of $\frac{-k}{24\pi}$ inches/week

(b) Does the radius change more quickly at the beginning of the life of the toilet paper roll or near the end (that is, when the radius is larger or smaller)? (3 points)

$$\frac{dr}{dt} = \frac{-k}{12\pi r(t)}.$$

When r is larger, $\left|\frac{-k}{12\pi r(t)}\right|$ is smaller and when r is smaller, $\left|\frac{-k}{12\pi r(t)}\right|$ is larger.

So near the beginning of the life of the roll of toilet paper, the radius decreases more slowly. Near the end of the life of the roll, the radius changes more quickly.

6. Production of toys at Santa's workshop follows the Cobb-Douglas production formula

$$P = 2L^{3/2}K$$

where P is the production level, L is the labor cost (in billions of dollars), and K is the cost of equipment (a.k.a capital, in billions of dollars). Suppose that this year, Santa must produce 2 billion toys for the good children of the world.

(a) At what rate can Santa replace labor with capital? That is, if labor changes a little, how must capital change in response to keep production levels at the target? (2 points)

The question is asking for $\frac{dK}{dL}$.

$$2 = 2L^{3/2}K$$

Use implicit differentiation:

$$2 = 2L^{3/2}K(L)$$

$$0 = 2\left[\frac{3}{2}L^{\frac{1}{2}}K(L) + L^{3/2}\frac{dK}{dL}\right]$$

So $$\frac{dK}{dL} = \frac{-\frac{3}{2}L^{\frac{1}{2}}K(L)}{L^{3/2}} = \frac{-3}{2}\frac{K}{L}$$

$$\left(\text{or } \frac{dK}{dL} = -\frac{3}{2}\cdot\frac{1}{L^{5/2}}\right).$$

(b) Suppose that labor cost is capped at 1 billion dollars (because of a shortage of skilled elves). How much should Santa spend on labor and on equipment to minimize his total cost, $L+K$? To solve this problem, are you optimizing over an open or closed interval? (9 points)

$$C = L + K = L + \frac{\cancel{2}\,1}{\cancel{2}L^{3/2}} = L + \frac{1}{L^{3/2}}.$$

To optimize: Find critical pts:

$$C'(L) = 1 + L^{-5/2}\cdot\left(-\frac{3}{2}\right) = 0 \text{ when}$$

$$\frac{3}{2}L^{-5/2} = 1, \quad \text{i.e. when } L = \left(\frac{2}{3}\right)^{-2/5} = \left(\frac{3}{2}\right)^{2/5}$$

We're optimizing over L in $[0,1]$, a closed interval.

$\left(\frac{3}{2}\right)^{2/5}$ = (something bigger than 1)$^{\text{positive power}}$ > 1, so our critical point isn't in the interval.

Check end points: $C = L + K$ when $L=0$: $C = 0 + \frac{1}{0}$ undefined (infinite).

when $L=1$: $C = 1 + \frac{1}{1} = \boxed{2}.$

Santa should spend 1 billion on labor and 1 billion on capital.

Formula Sheet

You may use these formulas when appropriate.

$\sin(\pi/2 - x) = \cos x$

$\cos(\pi/2 - x) = \sin x$

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\sin(x-y) = \sin x \cos y - \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\cos(x-y) = \cos x \cos y + \sin x \sin y$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Volume of a sphere $= \frac{4}{3}\pi r^3$

Surface area of a sphere $= 4\pi r^2$

Volume of a cylinder $= \pi r^2 h$

Surface area of a cylinder $= 2\pi r^2 + 2\pi rh$

Volume of a cone $= \frac{1}{3}\pi r^2 h$