MATH 100 – PRACTICE EXAM 2

Name: SOLUTIONS

Lecture:	MWF 12	MWF 1	MWF 3
Recitation:	T. Crawford Th 11	T. Crawford Th 1	T. Crawford Th 2
	G. Chiloyan Th 3	G. Chiloyan Th 11	G. Chiloyan Th 12

FOR FULL CREDIT, SHOW ALL WORK
NO CALCULATORS

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1. (a) **Using the limit definition of the derivative**, compute the derivative of $f(x) = \frac{1}{\sqrt{x}}$. (9 points)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x(x+h)}} = \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x(x+h)}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{h\sqrt{x(x+h)}\,(\sqrt{x} + \sqrt{x+h})} = \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}\sqrt{x(x+h)}\,(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x(x+h)}\,(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{\sqrt{x^2}\,(\sqrt{x} + \sqrt{x})}$$

$$= \boxed{\frac{-1}{2x\sqrt{x}}}$$

(b) Did you get the right answer in part (a)? How do you know? (2 points)

Yes. The power rule says $\frac{d}{dx} x^{-1/2} = \frac{-1}{2} x^{-3/2} = \frac{-1}{2x\sqrt{x}}$. ✓

2. Explain why there is a solution to the equation $e^x = x^2 - x^3$. *You must have at least one complete sentence in your answer for full credit.* (10 points)

We will use the intermediate value theorem.

Let $f(x) = e^x - (x^2 - x^3) = e^x - x^2 + x^3$.

To show that $e^x = x^2 - x^3$ has a solution, we need to show that $f(x) = 0$ for some x.

Note that $f(0) = e^0 - 0^2 + 0^2 = 1 > 0$

and $f(-1) = e^{-1} - (-1)^2 + (-1)^3 = \frac{1}{e} - 2 < 0$ (since $e \approx 2.718$, $\frac{1}{e}$ is definitely less than 1)

Also, $f(x)$ is a continuous function since it is the sum of an exponential function and a polynomial, both of which are continuous.

Therefore, the intermediate value theorem says $f(x) = 0$ for some x between -1 and 0, and $e^x = x^2 - x^3$ has a solution.

3. (a) Find the equation of a general tangent line to the graph of $y = x^2 + 2$. (Hint: pick a general point x_0, and find the equation of the tangent line to $(x_0, f(x_0))$ in terms of x_0, x and y.) (7 points)

$y' = 2x$, so at $x = x_0$, the slope of a tangent line is $2x_0$

Use point-slope form to get the equation of the tangent line:

$y - f(x_0) = 2x_0 (x - x_0)$

But $f(x_0) = x_0^2 + 2$, so

$$\boxed{y - (x_0^2 + 2) = 2x_0 (x - x_0)}$$

(b) Two different lines tangent to $y = x^2 + 2$ go through the point (0,1). Find the equations of both of those lines AND the points where these lines are tangent to the parabola. (3 points)

Two lines go through $(0, 1)$: Means $x = 0$, $y = 1$ in the equation of the line we found in part (a).

$1 - (x_0^2 + 2) = 2x_0 (-x_0)$

$1 - x_0^2 - 2 = -2x_0^2$

So $x_0^2 - 1 = 0 \Rightarrow x_0 = \pm 1$.

The equation of the first line is

$y - 3 = 2(x - 1)$ and is tangent at $(1, 3)$

The equation of the second line is

$y - 3 = -2(x + 1)$ and is tangent at $(-1, 3)$.

4. In terms of $g(x)$ and $g'(x)$, find the derivative of $y = \dfrac{1}{\sin(g(x))}$. (10 points)

Use the quotient rule and chain rule.

$$y' = \frac{\sin(g(x)) \cdot (0) - 1\left(\frac{d}{dx}\sin(g(x))\right)}{\sin^2(g(x))}$$

$$= \boxed{\frac{-\cos(g(x)) \cdot g'(x)}{\sin^2(g(x))}}$$

(Alternately, say $y = [\sin(g(x))]^{-1}$ and use the chain rule twice:

$$y' = -1\,[\sin(g(x))]^{-2} \cdot \frac{d}{dx}\sin(g(x))$$

$$= -\frac{1}{\sin^2(g(x))} \cdot \cos(g(x))\; g'(x).\ )$$

5. (a) Find $f'(x)$, $f''(x)$, and $f'''(x)$ for the function $f(x) = xe^{-x}$. (5 points)

$$f'(x) = -xe^{-x} + e^{-x}$$

$$f''(x) = xe^{-x} + - e^{-x} - e^{-x} = xe^{-x} - 2e^{-x}$$

$$f'''(x) = -xe^{-x} + e^{-x} + 2e^{-x} = -xe^{-x} + 3e^{-x}.$$

(b) Give a general formula for $f^{(n)}(x)$. You don't need to prove your formula, but your answer should be in terms of x and n. (5 points)

If the pattern isn't clear yet, try

$$f^{(4)}(x) = xe^{-x} - e^{-x} - 3e^{-x} = xe^{-x} - 4e^{-x}.$$

$f^{(n)}(x) = xe^{-x} - ne^{-x}$ if n is even

$f^{(n)}(x) = -xe^{-x} + ne^{-x}$ if n is odd

(or $f^{(n)}(x) = (-1)^n xe^{-x} + (-1)^{n+1} ne^{-x}$).

6. Compute the following derivatives, showing all steps. You do not need to use the limit definition of the derivative or state which rules you are using. (3 points each)

(a) $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$y' = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{\cancel{e^{2x}} + \cancel{e^{-2x}} + 2 - (\cancel{e^{2x}} + \cancel{e^{-2x}} - 2)}{e^{2x} + e^{-2x} + 2}$$

$$= \boxed{\frac{4}{(e^x + e^{-x})^2}}$$

(b) $y = x^{2^x}$ (Hint: Rewrite this function as a power of e.)

$$y = \left(e^{\ln x}\right)^{2^x} = e^{2^x \cdot \ln x}$$

$$y' = e^{2^x \ln x} \cdot \left(\frac{d}{dx}\left[2^x \ln x\right]\right)$$

$$= e^{2^x \ln x} \cdot \left(\ln 2 \cdot 2^x \ln x + \frac{2^x}{x}\right)$$

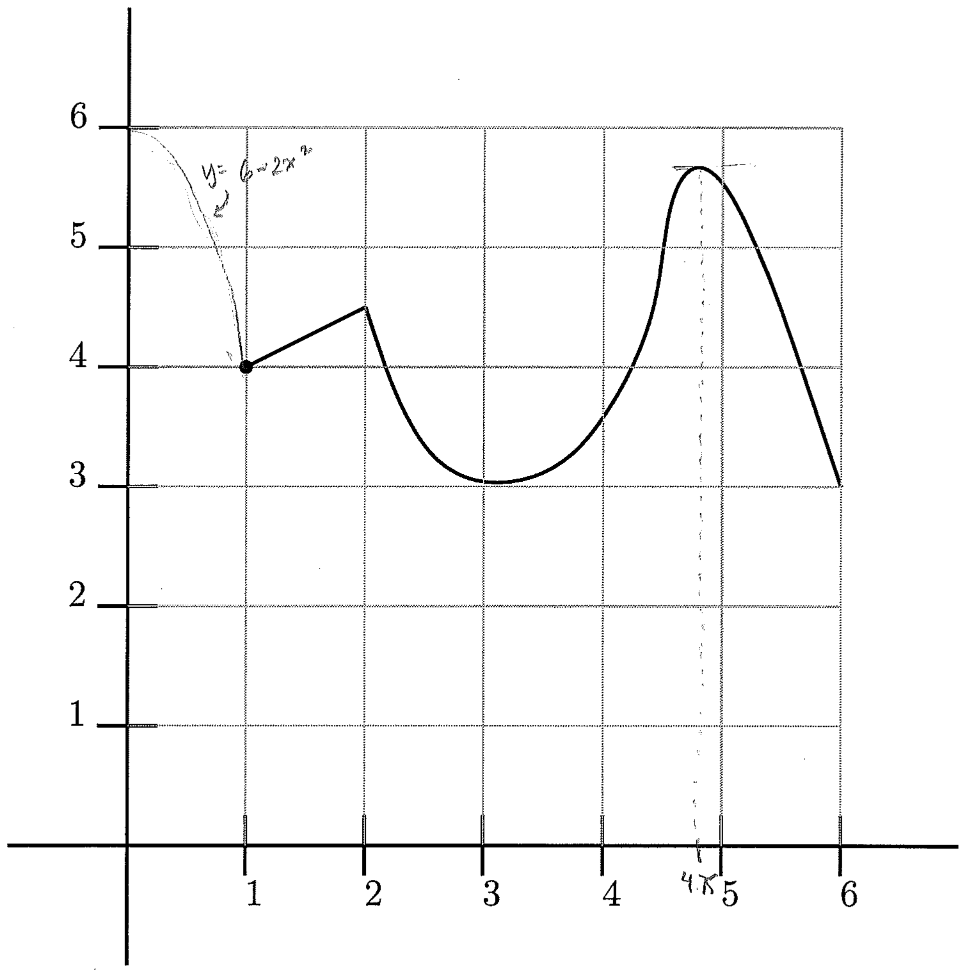
$$= \boxed{2^x x^{2^x} \cdot \left(\ln 2 \cdot \ln x + \frac{1}{x}\right)}$$

(c) $y = \sqrt{1 + \sin x \cos x}$

$$y' = \frac{1}{2}\left(1 + \sin x \cos x\right)^{-\frac{1}{2}} \cdot \left(\cos^2 x + -\sin^2 x\right)$$

$$= \boxed{\frac{\cos^2 x - \sin^2 x}{2\sqrt{1 + \sin x \cos x}}}$$

7. Below is the portion of the graph of a function $f(x)$. (It is the portion where $1 \leq x \leq 6$.) Use this graph to answer the following questions.



(a) If $f(x)$ is defined by $f(x) = 6 - cx^2$ for $0 \leq x < 1$, what value of c will make f continuous? (5 points)

$6 - c = 4 \Rightarrow \boxed{c = 2.}$

(b) For what values of x is $f'(x) = 0$? (3 points)

$f'(x) = 0$ for $x = 3$ and $x \approx 4.75$

(c) For what value of x does $f'(x)$ take its maximum value on the range $0 < x < 6$? (2 points)

$f'(x)$ takes its maximum value at about $4.5 = x$.

(d) Name all values of x in the range $0 < x < 6$ where the derivative does not exist. (5 points)

The derivative does n't exist at

$x = 1$ and $x = 2$.