MATH 100 – MIDTERM 3

Name: SOLUTIONS

Lecture: MWF 12 MWF 1 MWF 3

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FOR FULL CREDIT, SHOW ALL WORK
NO CALCULATORS

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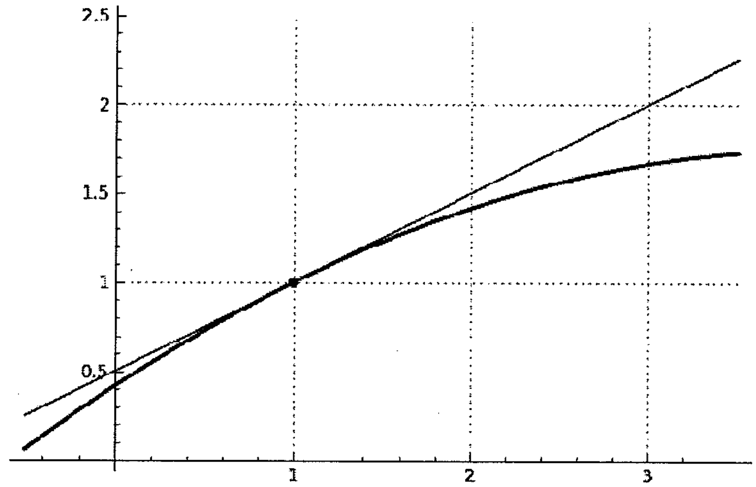
1. The graph of $f(x)$ and the line tangent to the graph at (1,1) is shown below. What is the equation of the line tangent to the graph of $f^{-1}(x)$ at (1,1)? (10 points)



Use: $g'(b) = \dfrac{1}{f'(g(b))}$ where g, f are inverse functions.

$f'(1) = \frac{1}{2}$, so $g'(1) = \dfrac{1}{f'(g(1))} = \dfrac{1}{f'(1)} = \dfrac{1}{1/2} = 2$.

The equation of the line tangent to $f^{-1}(x)$ at $(1,1)$ is $\boxed{y-1 = 2(x-1)}$

2. (a) Find the equation of the line tangent to the curve defined by the equation

$$x + 2^x y + y^2 = 2$$

at the point $(2,0)$. (5 points)

Use implicit differentiation to find $\frac{dy}{dx}$.

$$1 + 2^x \frac{dy}{dx} + (\ln 2)\, 2^x y + 2y \frac{dy}{dx} = 0$$

At $(2,0)$, $x = 2$, $y = 0$. Plug in to get

$$1 + 2^2 \frac{dy}{dx} + \underline{(\ln 2)\, 2^2 \cdot 0} + 2 \cdot 0 \frac{dy}{dx} = 0$$

$$1 + 4 \frac{dy}{dx} = 0$$

So $\frac{dy}{dx} = \frac{-1}{4}$.

The equation of the line tangent at $(2,0)$ is

$$\boxed{y = \frac{-1}{4}(x-2).}$$

(b) Estimate the y-coordinate of the point on the curve with x-coordinate 2.1. (5 points)

Use linear approximation:

$$y \approx \frac{-1}{4}(2.1 - 2) = \frac{-1}{4}(0.1)$$

$$= \boxed{\frac{-1}{40}}$$

3. Benjamin is a very bad student and has put no work into his calculus class. He sets aside 12 hours to study the day before the final. He knows that the more he studies, the better he will do up to a point, but after a certain point, he will just tire himself out and his grade will go down. Suppose you can predict his grade by the formula

$$c = \frac{-1}{4}t^3 + 3t^2$$

where t is the number of hours he studies calculus, and c is the percentage he gets on the final exam.

(a) Since you are good at calculus, Benjamin has asked you to tell him how long he should study to get the best possible score and what that score is. What do you tell him? (Besides the fact that this is a horrible plan.) (5 points)

$c' = -\frac{3}{4}t^2 + 6t$

Find critical points: $c' = 0$ when $-\frac{3}{4}t^2 + 6t = t(-\frac{3}{4}t + 6) = 0$

i.e. when $t = 0$ or when $t = \frac{-6}{-3/4} = 8$.

Test end points and critical points.

$c(0) = 0$

$c(8) = -\frac{1}{4}8^3 + 3 \cdot 8^2 = 8^2(-\frac{1}{4} \cdot 8 + 3) = 64(-2+3) = 64$.

$c(12) = -\frac{1}{4}(12)^3 + 3(12^2) = 12^2(-\frac{1}{4} \cdot 12 + 3) = 0$

Benjamin should study for 8 hours. He will get a 64 on the exam.

(b) Now suppose Benjamin has also failed to prepare for his philosophy final, which is on the same day as his calculus final. Suppose he can predict his philosophy grade with the formula

$$p = \frac{-1}{2}(s-12)^2 + 72$$

where s is the number of hours he studies philosophy, and p is the percentage he gets on the philosophy final.

Advise Benjamin on how to split his 12 hours of studying between philosophy and calculus. (Note: If he spends t hours on calculus, he will spend $12-t$ hours on philosophy.) How many hours should he spend on each to maximize the sum of his grades, $p+c$? (5 points)

If he studies t hours for calculus and $12-t$ hours for philosophy,

$p + c = -\frac{1}{4}t^3 + 3t^2 + -\frac{1}{2}((12-t) - 12)^2 + 72$

$= -\frac{1}{4}t^3 + 3t^2 - \frac{1}{2}t^2 + 72$

$(p+c)' = -\frac{3}{4}t^2 + 6t - t = -\frac{3}{4}t^2 + 5t$

Find critical points: $(p+c)' = 0$ when $-\frac{3}{4}t^2 + 5t = t(-\frac{3}{4}t + 5) = 0$.

i.e. when $t = 0$ or when $t = \frac{-5}{-3/4} = \frac{20}{3}$.

Test end points and critical points.

$(p+c)(0) = 72$

$(p+c)(12) = 12^2(-\frac{1}{4} \cdot 12 + 3 - \frac{1}{2}) + 72 = 144(-3+3-\frac{1}{2}) + 72 = -72 + 72 = 0$

cont →

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Test endpts and critical pts (cont)

$$(p+c)\left(\tfrac{20}{3}\right) = \left(\tfrac{20}{3}\right)^2\left(-\tfrac{1}{\cancel{4}}\cdot\tfrac{\overset{5}{\cancel{20}}}{3}+3-\tfrac{1}{2}\right)+72$$

$$= \left(\tfrac{20}{3}\right)^2\left(-\tfrac{5}{3}+3-\tfrac{1}{2}\right)+72$$

$$= \left(\tfrac{20}{3}\right)^2\left(-\tfrac{10}{6}+\tfrac{18}{6}-\tfrac{3}{6}\right)+72$$

$$= \left(\tfrac{20}{3}\right)^2\left(\tfrac{5}{6}\right)+72 > 72 \text{ since } \left(\tfrac{20}{3}\right)^2\left(\tfrac{5}{6}\right)>0.$$

He should spend $\frac{20}{3}$ hours on calculus and $12-\frac{20}{3}=\frac{16}{3}$ hours on philosophy.

4. Let $f(x) = \sin^2 x$.

(a) Find all local minima and maxima of f on the interval $(0, 2\pi)$. (3 points)

$f'(x) = 2\sin x \cos x$ (Chain rule)

$f'(x) = 0$ when $\sin x = 0$ or when $\cos x = 0$, i.e. when

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

	0		$\frac{\pi}{2}$		π		$\frac{3\pi}{2}$		2π
$\sin x$		$+$		$+$		$-$		$-$	
$\cos x$		$+$		$-$		$-$		$+$	
$2\sin x\cos x$		$+$		$-$		$+$		$-$	
			↑ max		↑ min		↑ max		

$f(x)$ has a local minimum at $x = \pi$, and the value is $\sin^2(\pi) = 0$

$f(x)$ has local maxima at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

The values are $f(\frac{\pi}{2}) = \sin^2(\frac{\pi}{2}) = 1^2 = 1$

and $f(\frac{3\pi}{2}) = \sin^2(\frac{3\pi}{2}) = (-1)^2 = 1$.

(b) Find the intervals in $(0, 2\pi)$ on which f is increasing and the intervals in $(0, 2\pi)$ on which f is decreasing. (2 points)

From above: f is increasing on $(0, \frac{\pi}{2})$ and $(\pi, \frac{3\pi}{2})$

(when $f' > 0$)

f is decreasing on $(\frac{\pi}{2}, \pi)$ and $(\frac{3\pi}{2}, 2\pi)$

(when $f' < 0$).

Note: For parts (c) and (d), $f(x)$ is still $\sin^2 x$.

(c) Find the intervals in $(0, 2\pi)$ on which f is concave up and the intervals in $(0, 2\pi)$ on which f is concave down. (3 points)

$$f''(x) = \frac{d}{dx}\left(2\sin x \cos x\right) = 2\left(\cos x \cdot \cos x + (-\sin x)(\sin x)\right)$$
$$= 2\left(\cos^2 x - \sin^2 x\right) = 2 \cdot \cos 2x$$

$f''(x) = 0$ when $\cos^2 x = \sin^2 x$, i.e. when $\cos x = \pm \sin x$.

$f''(x) = 0$ when $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

0 to $\frac{\pi}{4}$	$\frac{\pi}{4}$ to $\frac{3\pi}{4}$	$\frac{3\pi}{4}$ to $\frac{5\pi}{4}$	$\frac{5\pi}{4}$ to $\frac{7\pi}{4}$	$\frac{7\pi}{4}$ to 2π
$2\cos(2\cdot 0) = 2\cos 0 = 2$	$2\cos(2\frac{\pi}{2}) = 2\cos\pi = -2$	$2\cos(2\pi) = 2$	$2\cos(\frac{3\pi}{2}\cdot 2) = 2\cos(3\pi) = -2$	$2\cos(2\cdot 2\pi) = 2\cos(4\pi) = 2$
$+$	$-$	$+$	$-$	$+$

f is concave up on $(0, \frac{\pi}{4})$, $(\frac{3\pi}{4}, \frac{5\pi}{4})$, and $(\frac{7\pi}{4}, 2\pi)$ (when $f'' > 0$).

f is concave down on $(\frac{\pi}{4}, \frac{3\pi}{4})$ and $(\frac{5\pi}{4}, \frac{7\pi}{4})$ (when $f'' < 0$).

(d) Find all points of inflection of f in $(0, 2\pi)$. (2 points)

Points of inflection occur when $f'' = 0$ and f'' changes sign.

Points of inflection are at

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

5. Let $f(x) = x^x$. Show that $x = \frac{1}{e}$ is a critical point of f. (10 points)

Use logarithmic differentiation.

$$\ln f(x) = \ln x^x = x \ln x.$$

$$(\ln f(x))' = \frac{f'(x)}{f(x)} = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$$

So $f'(x) = (1 + \ln x) x^x$.

To show $x = \frac{1}{e}$ is a critical point:

$$f'(\tfrac{1}{e}) = (1 + \ln \tfrac{1}{e}) (\tfrac{1}{e})^{1/e}$$

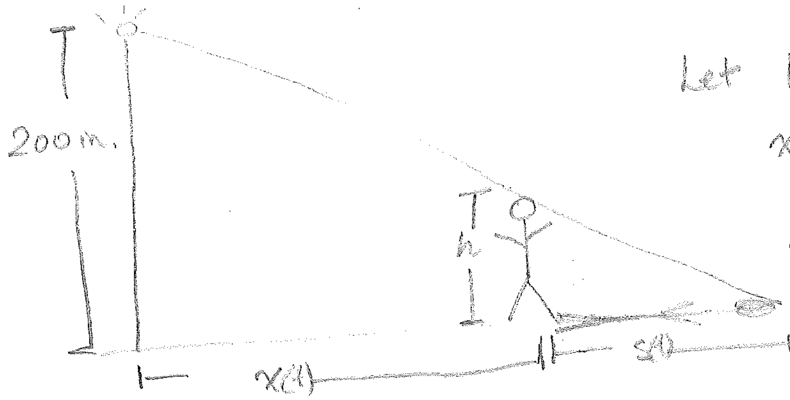
$$= (1 + -1) (\tfrac{1}{e})^{1/e}$$

$$= 0 \cdot (\tfrac{1}{e})^{1/e}$$

$$= 0.$$

Since $f'(\frac{1}{e}) = 0$, $\frac{1}{e}$ is a critical point of f.

6. Dustin walks away from a 200 inch tall street light at a speed of 50 inches per second. If Dustin's shadow is growing at a rate of 30 inches per second, how tall is he? (10 points)



Let h = Dustin's height (constant)

$x(t)$ = Dustin's distance from the street light at time t.

$s(t)$ = the length of Dustin's shadow at time t.

Relate the variables: $\frac{s(t)}{h} = \frac{s(t) + x(t)}{200}$ (similar triangles).

Know: $\frac{dx}{dt} = 50$ in/sec and $\frac{ds}{dt} = 30$ in/sec.

Differentiate to relate the rates:

$$\frac{1}{h}\frac{ds}{dt} = \frac{1}{200}\left(\frac{ds}{dt} + \frac{dx}{dt}\right)$$

Plug in: $\frac{1}{h} \cdot 30 = \frac{1}{200}(30 + 50)$

So $h = \frac{\overset{25}{\cancel{200}} \cdot 3\cancel{0}}{\cancel{80}} = 25 \cdot 3 =$ $\boxed{75 \text{ inches tall}}$

Formula Sheet

You may use these formulas when appropriate.

$\sin(\pi/2 - x) = \cos x$

$\cos(\pi/2 - x) = \sin x$

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\sin(x-y) = \sin x \cos y - \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\cos(x-y) = \cos x \cos y + \sin x \sin y$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Volume of a sphere $= \frac{4}{3}\pi r^3$

Surface area of a sphere $= 4\pi r^2$

Volume of a cylinder $= \pi r^2 h$

Surface area of a cylinder $= 2\pi r^2 + 2\pi rh$

Volume of a cone $= \frac{1}{3}\pi r^2 h$