MATH 100 – MIDTERM 2

Name: SOLUTIONS

Lecture:	MWF 12	MWF 1	MWF 3
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FOR FULL CREDIT, SHOW ALL WORK
NO CALCULATORS

1	
2	
3	
4	
5	
6	

1. Compute the following limits. *Reminder: For full credit, show all work.* (4 points each)

(a) $\lim_{x\to\infty} \frac{x^2+3x+2}{\sqrt{4x^4-4x^2+1}}$

$$\lim_{x\to\infty} \frac{x^2+3x+2}{\sqrt{4x^4-4x^2+1}} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \cdot \frac{x^2+3x+2}{\sqrt{4x^4-4x^2+1}}$$

$$= \lim_{x\to\infty} \frac{1+\frac{3}{x}+\frac{2}{x^2}}{\sqrt{4-\frac{4}{x^2}+\frac{1}{x^4}}} = \frac{1+0+0}{\sqrt{4-0+0}} = \boxed{\frac{1}{2}}$$

(b) $\lim_{x\to 1} \frac{1-x^2}{x^3+x-1}$

$\frac{1-x^2}{x^3+x-1}$ is continuous at $x=1$, so just plug in.

$$\lim_{x\to 1} \frac{1-x^2}{x^3+x-1} = \frac{1-1}{1+1-1} = \frac{0}{1} = \boxed{0}$$

(c) $\lim_{x\to a} \frac{\frac{1}{x}-\frac{1}{a}}{x-a}$ (Hint: Here, a is a constant.)

$$\lim_{x\to a} \frac{\frac{1}{x}-\frac{1}{a}}{x-a} = \lim_{x\to a} \frac{\frac{a-x}{ax}}{x-a} = \lim_{x\to a} \frac{-(\cancel{x-a})}{(\cancel{x-a})\cdot ax}$$

$$= \lim_{x\to a} \frac{-1}{ax} = \boxed{\frac{-1}{a^2}}$$

2. Let $f(x) = \dfrac{x}{x^2 g(x)+1}$, and suppose $g(1) = 7$ and $g'(1) = 2$.

What is the equation of the line tangent to the graph of $f(x)$ at $x = 1$? (9 points)

To get the slope of the tangent line, we need $f'(1)$.

$$f'(x) = \frac{(x^2 g(x)+1)\,1 - x\left(2x\,g(x) + x^2 g'(x)\right)}{\left(x^2 g(x)+1\right)^2}$$

So $$f'(1) = \frac{(1^2\cdot 7+1)\cdot 1 - 1\cdot(2\cdot 1\cdot 7 + 1\cdot 2)}{(1^2\cdot 7+1)^2}$$

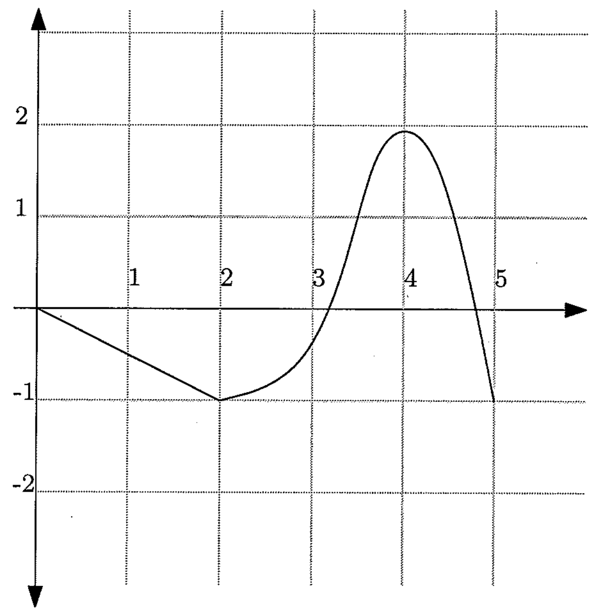
$$= \frac{8-16}{8^2} = \frac{-8}{8^2} = \frac{-1}{8}$$

Also, $$f(1) = \frac{1}{1^2 g(1)+1} = \frac{1}{7+1} = \frac{1}{8}.$$

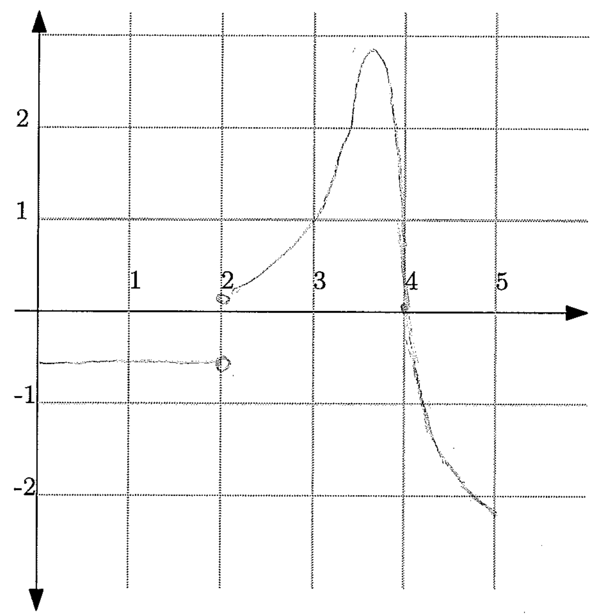
So the tangent line goes through $(1, \frac{1}{8})$ and has slope $\frac{-1}{8}$.

$\boxed{y - \frac{1}{8} = \frac{-1}{8}(x-1)}$ (or $y = \frac{-1}{8}x + \frac{1}{4}$).

3. The graph of a function $f(x)$ is shown below.



Sketch the graph of $f'(x)$. (8 points)



4. Using that $\cos(\arccos x) = x$, show that the derivative of $\arccos x$ is $\frac{-1}{\sqrt{1-x^2}}$. (10 points)

Take the derivative of both sides:

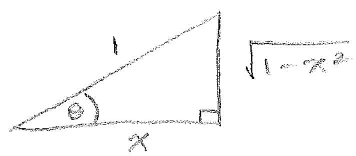
$$\frac{d}{dx} \cos(\arccos x) = \frac{d}{dx} x$$

$$-\sin(\arccos x) \cdot \frac{d}{dx} \arccos x = 1$$

So $\frac{d}{dx} \arccos x = \frac{-1}{\sin(\arccos x)}$

But $\sin(\arccos x)$ can be simplified.

If $\arccos x = \theta$, $\cos\theta = x$.



So $\sin\theta = \sin(\arccos x) = \sqrt{1-x^2}$.

Thus, $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$

5. A car accelerates for 2 minutes, then brakes suddenly when the driver notices a traffic jam ahead. Suppose the position function is

$$s(t) = \begin{cases} t^2 & 0 \le t < 2 \\ A + Bt - t^2 & 2 \le t \le 4 \end{cases}$$

Assuming both the position function and velocity function are continuous, find A and B. (10 points)

For position to be continuous, we only need to check at $t = 2$:

$4 = A + 2B - 4 \Rightarrow$ Need $A + 2B = 8$

For velocity to be continuous, take derivatives.

$$s'(t) = \text{velocity}(t) = \begin{cases} 2t & 0 \le t < 2 \\ B - 2t & 2 \le t \le 4. \end{cases}$$

Need $2(2) = B - 2 \cdot 2 \Rightarrow B = 8.$

So $A + 16 = 8 \Rightarrow A = -8.$

So $\boxed{\begin{matrix} A = -8 \\ B = 8 \end{matrix}}$

6. (a) **Using the limit definition of the derivative**, compute

$$\frac{d}{dx}\cos x.$$

See the last page of the exam for some trig identities which you may find helpful. (10 points)

$$\frac{d}{dx}\cos x = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \cos x \left(\frac{\cos h - 1}{h}\right) - \sin x \left(\frac{\sin h}{h}\right)$$

$$= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= (\cos x)(0) - (\sin x)(1)$$

$$= -\sin x. \checkmark$$

(b) Did you get the right answer in part (a)? How do you know? (1 point)

I know that $\frac{d}{dx}\cos x = -\sin x$, so yes, I got the right answer.

Trig Formula Sheet

You may use these formulas when appropriate.

$\sin(\pi/2 - x) = \cos x$

$\cos(\pi/2 - x) = \sin x$

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\sin(x-y) = \sin x \cos y - \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\cos(x-y) = \cos x \cos y + \sin x \sin y$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$