MATH 100 – MIDTERM 1

Name: SOLUTIONS

Lecture:	MWF 12	MWF 1	MWF 3
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FOR FULL CREDIT, SHOW ALL WORK
NO CALCULATORS

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1. (a) Find the domain and range of the following function. (5 points)

$$f(x) = 1 - x^2$$

Domain: all real numbers. (f is a polynomial, and any real number can be plugged in.)

Range: We know $x^2 \geq 0$, so $1 - x^2 \leq 1$.
Therefore, the range is real numbers less than or equal to 1

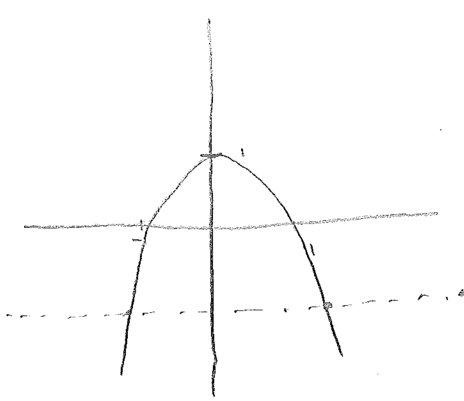
DOMAIN: $\mathbb{R}$ RANGE: $(-\infty, 1]$

(b) Explain why f is not one-to-one. (3 points)

Two possible solutions:

1) f is not one-to-one because there are two different inputs that have the same output. For example, $f(1) = f(-1) = 0$.

2) f is not one-to-one because the graph fails the horizontal line test.



horizontal line crosses the graph at two points.

Note: For parts (c) and (d), $f(x)$ is still
$$f(x) = 1 - x^2.$$

(c) Given that f is one-to-one on the domain $[0, \infty)$, find f^{-1} on this restricted domain. (5 points)

$y = 1 - x^2$

$x^2 = 1 - y$

$x = \pm\sqrt{1-y}$

Since x is positive in this restricted domain, choose

$x = \sqrt{1-y}$.

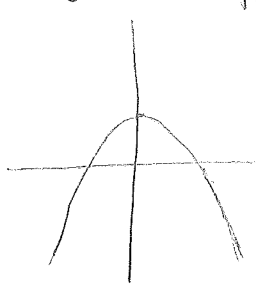
ANSWER: $f^{-1}(y) = \sqrt{1-y}$

(d) Did you get the right domain and range in part (a)? How can you tell? (2 points)

Two possible solutions:

1) The domain of f^{-1} is $y \leq 1$ since we need $1-y \geq 0$ to take the square root. The domain of f^{-1} should be the range of f, and in my answer this is the case, so I got the range of f right. The domain is correct because any real number can be squared and subtracted from 1.

2) I know the graph of $y = x^2$, and the graph of f is just flipped vertically and shifted up one unit.

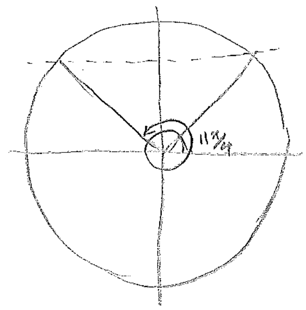


We can see from the graph that $y = 1$ is the maximum y value, and any value can be plugged in for x. Thus, the domain is $\mathbb{R}$ and the range is $(-\infty, 1]$, as I said in part (a).

2. (a) Compute

$$\arcsin(\sin \tfrac{11\pi}{4}).$$

(5 points)



$\frac{11\pi}{4}$ is in the 2^{ND} quadrant

$\sin \frac{11\pi}{4} = \frac{\sqrt{2}}{2}$

We want an angle θ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ such that $\sin\theta = \frac{\sqrt{2}}{2}$.

That angle is $\frac{\pi}{4}$

ANSWER: $\frac{\pi}{4}$

(b) Compute

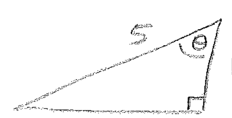
$$\cos(\arccos \tfrac{1}{5}).$$

(5 points)

$\arccos \frac{1}{5} = \theta$ means $\cos\theta = \frac{1}{5}$ and $0 \le \theta \le \pi$.

So $\cos\theta = \frac{1}{5}$.

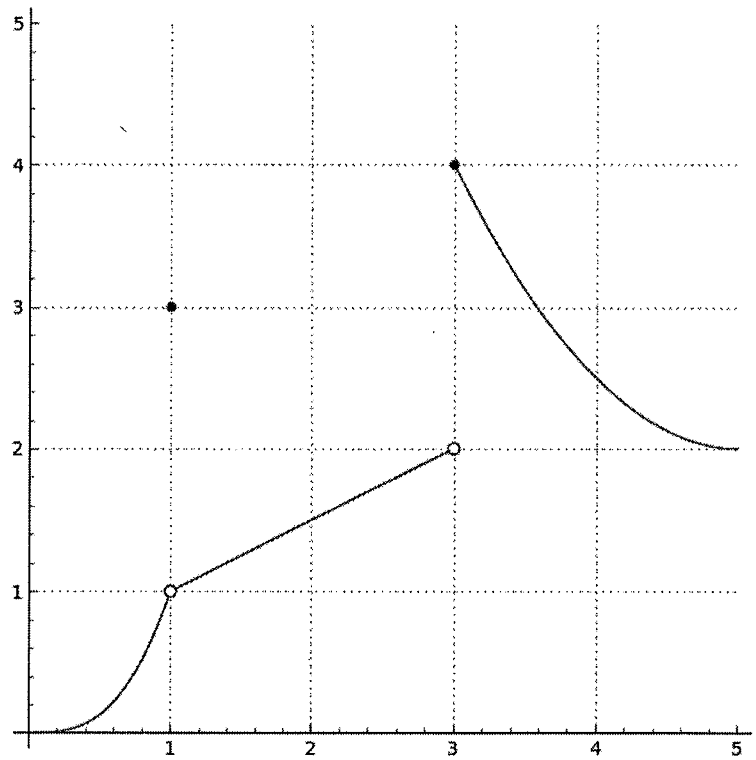
(Alternately, draw a triangle.



$\theta = \arccos \frac{1}{5}$.
)

ANSWER: $\frac{1}{5}$.

3. Use the given graph of $f(x)$ to answer the following questions. *You need not show work for this problem, but wrong answers with no work will be given no credit.* (2 points each)



(a) What is $\lim_{x\to 2^+} f(x)$?

ANSWER: 1.5

(b) Name a point c where $\lim_{x\to c} f(x)$ exists, but is not equal to $f(c)$

ANSWER: $c = 1$ $\left(\lim_{x\to 1} f(x) = 1, \quad f(1) = 3 \right)$.

(c) Name a point c where $\lim_{x\to c} f(x)$ does not exist.

ANSWER: $c = 3$ $\left(\lim_{x\to 3^-} f(x) = 2, \quad \lim_{x\to 3^+} f(x) = 4 \right)$

(d) Name a point c where $\lim_{x\to c^-} f(x) = f(c)$.

ANSWER: Many possibilities, e.g. $c = 4$

(e) Assuming that the domain of $f(x)$ is all real numbers, and this is only a piece of the graph, can you say whether $\lim_{x\to 5} f(x)$ exists?

Circle one: YES NO

(Graph could look like

)

4. Solve for x. (10 points)

$$2 \cdot (5^{x \log_5 2}) + \log_3 \frac{1}{9} = 0$$

$\log_3 \frac{1}{9} = -2$ since $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

$5^{x \log_5 2} = (5^{\log_5 2})^x = 2^x$.

So we have $2 \cdot 2^x + -2 = 0$

$2 \cdot 2^x = 2$

$2^x = 1$

So $\boxed{x = 0}$

5. Which of the following limits can be solved with only the basic limit laws? (In other words, to which of the following limits do the basic limit laws apply?) ***You DO NOT need to compute the limits.*** *You must give at least a few words of explanation for full credit. Correct answers with no work will receive 2 points out of 3.*

(3 points each)

(a) $\lim\limits_{x\to\pi} \dfrac{\sin x}{x}$

We can make this $\dfrac{\lim\limits_{x\to\pi} \sin x}{\lim\limits_{x\to\pi} x}$ so long as the limits in the numerator and denominator exist and the limit in the denominator is not 0. In this case, both limits exist and the limit in the denominator is $\pi \neq 0$.

Circle one: Basic limit laws DO APPLY DO NOT APPLY

(b) $\lim\limits_{x\to 2} \sqrt{3x^2+6}$

We can apply basic limit laws as long as all the limits we get when we simplify exist (i.e. are numbers), which they are in this case.

Circle one: Basic limit laws DO APPLY DO NOT APPLY

(c) $\lim\limits_{x\to 3} \dfrac{x}{x^2-9}$

The first step in applying the basic limit laws would be to split this into

$\dfrac{\lim\limits_{x\to 3} x}{\lim\limits_{x\to 3} x^2-9}$, but we can only do this if both the limits exist and the limit of the denominator is not 0. $\lim\limits_{x\to 3} x^2-9=0$, so we can't use this limit law.

Circle one: Basic limit laws DO APPLY DO NOT APPLY

6. Define

$$L(a) = \lim_{x\to 0} \frac{a^x - 1}{x}$$

for any real number $a > 0$. Show that $L(ab) = L(a) + L(b)$ for a and b greater than 0 using the basic limit laws and the following givens. (6 points)

- $\lim_{x\to 0} \frac{a^x - 1}{x}$ exists for any $a > 0$.
- $\lim_{x\to 0} a^x = 1$ for any $a > 0$.
- You can and should use the algebraic manipulation

$$\frac{a^x b^x - 1}{x} = \frac{a^x(b^x - 1)}{x} + \frac{a^x - 1}{x}.$$

(A hint to get you started: Write down what $L(ab)$, $L(a)$, and $L(b)$ are in terms of limits by substituting into the formula given.)

$$L(ab) = \lim_{x\to 0} \frac{(ab)^x - 1}{x} = \lim_{x\to 0} \frac{a^x b^x - 1}{x}$$

$$= \lim_{x\to 0} \left(\frac{a^x(b^x - 1)}{x} + \frac{a^x - 1}{x} \right)$$

$$= \lim_{x\to 0} \frac{a^x(b^x-1)}{x} + \lim_{x\to 0} \frac{a^x - 1}{x}$$ (by basic limit laws)

$$= \left(\lim_{x\to 0} a^x\right)\left(\lim_{x\to 0} \frac{b^x - 1}{x}\right) + L(a)$$ (basic limit laws and the definition of $L(a)$)

$$= 1 \cdot L(b) + L(a).$$ (since $\lim_{x\to 0} a^x = 1$)

So $L(ab) = L(a) + L(b)$.