Homework 8 (Due 11/1)

Solutions

Section 3.7

73. $\frac{d^2}{dx^2}\sin(x^2) = \frac{d}{dx}\left[\cos(x^2)\cdot 2x\right]$

$= 2\cos(x^2) + 2x\left(-\sin(x^2)\cdot 2x\right)$

$= \boxed{2\cos(x^2) - 4x^2\sin(x^2)}$

78. $P = Ri^2$

$R = 1000\,\Omega$, $i = \sin(4\pi t)$.

$P = 1000\sin^2(4\pi t)$

$\frac{dP}{dt} = 1000\left(2\sin(4\pi t)\cdot\cos(4\pi t)\cdot 4\pi\right)$

$= 8000\pi\sin(4\pi t)\cos(4\pi t)$

$\left.\frac{dP}{dt}\right|_{t=1/3} = 8000\pi\sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{4\pi}{3}\right)$

$= 8000\pi\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{-1}{2}\right)$

$= \boxed{2000\sqrt{3}\,\pi}$

80. $L(t) = 32(1-e^{-0.37t})$

a) avg. length is changing at a rate of $L'(6)$ at age $t=6$ years.

$L'(t) = 32(0 - e^{-0.37t}(-0.37))$

$= (32)(0.37)e^{-0.37t}$

$L'(6) = (32)(0.37)e^{(-0.37)(6)} \approx \boxed{1.29 \text{ cm/year}}$

b) $L'(t) = 5$ for what value of t?

$(32)(0.37)e^{-0.37t} = 5$

$e^{-0.37t} = \frac{5}{(32)(0.37)}$

$-0.37t = \ln\left(\frac{5}{(32)(0.37)}\right)$

$t = -0.37\ln\left(\frac{5}{32\cdot 0.37}\right) \approx \boxed{6.32 \text{ years}}$

82. $M(t) = \left(a + (b-a)e^{kmt}\right)^{1/m}$

$$M'(t) = \frac{1}{m}\left(a + (b-a)e^{kmt}\right)^{1/m - 1} \cdot \left(0 + (b-a)e^{kmt} \cdot km\right)$$

$$= \left(a + (b-a)e^{kmt}\right)^{1/m - 1}(b-a)ke^{kmt}$$

$$M'(0) = \left(a + (b-a)e^{0}\right)^{1/m - 1}(b-a)ke^{0}$$

$$= (\cancel{a} + b - \cancel{a})^{1/m - 1}(b-a)k$$

$$= \boxed{kb^{1/m - 1}(b-a)}$$

Given:

84. $f(0) = 2$ $f'(0) = 3$, $h(0) = -1$, $h'(0) = 7$

a) $\frac{d}{dx}(f(x))^3 = 3(f(x))^2 \cdot f'(x)$

So the derivative at $x = 0$ is $3(f(0))^2 \cdot f'(0)$

$$= 3 \cdot 4 \cdot 3 = \boxed{36}$$

b) $\frac{d}{dx}f(7x) = 7 \cdot f'(7x)$

So the derivative at 0 is $7 \cdot f'(7 \cdot 0) = 7 \cdot f'(0) = 14.$

c) $\frac{d}{dx} f(4x)h(5x) = \left[\frac{d}{dx}f(4x)\right]h(5x) + f(4x)\left[\frac{d}{dx}h(5x)\right]$

$$= 4f'(4x) \cdot h(5x) + f(4x)(5 \cdot h'(5x))$$

So the derivative at 0 is $4 \cdot f'(0)h(0) + 5f(0)h'(0)$

$$= 4 \cdot 3 \cdot (-1) + 5 \cdot 2 \cdot 7$$

$$= -12 + 70 = \boxed{58}$$

94. $R = \sigma T^4$

The thing to notice here is that R and T are functions of t.

$R(t) = \sigma(T(t))^4$

So $\frac{dR}{dt} = \sigma \cdot 4T(t)^3 \cdot T'(t)$

$$= 4\sigma(T(t))^3 \cdot \frac{dT}{dt}$$

$$= 4(5.67 \times 10^{-8})(283)^3 \cdot 0.05 \approx 0.257$$

The units are: $(Js^{-1}m^{-2}\cancel{K^{-4}})(\cancel{K})^3 \cdot (\cancel{K}/yr) = Js^{-1}m^{-2}/yr.$

So $\frac{dR}{dt} = \boxed{0.257\ Js^{-1}m^{-2}/yr}$

Section 3.8

Preliminary questions

1. The slope of the line obtained by reflecting the line $y=\frac{x}{2}$ through $y=x$ is $\frac{1}{\frac{1}{2}} = \boxed{2}$

Exercises

1. $g(x)$ is the inverse of $f(x)=\sqrt{x^2+9}$ $(x \geq 0)$

$y=\sqrt{x^2+9}$

$y^2 = x^2+9 \rightsquigarrow x=\sqrt{y^2-9}$

$g(x)=\sqrt{x^2-9}$.

① From thm 1, $g'(x) = \dfrac{1}{f'(g(x))}$

$f'(x) = \frac{1}{2}(x^2+9)^{-1/2} \cdot 2x$

So $g'(x) = \dfrac{1}{\dfrac{\sqrt{x^2-9}}{\left(\sqrt{x^2-9}^{\,2}+9\right)^{1/2}}} = \dfrac{1}{\dfrac{\sqrt{x^2-9}}{(x^2-9+9)^{1/2}}}$

$= \boxed{\dfrac{x}{\sqrt{x^2-9}}}$

② Direct calculation:

$g(x)=\sqrt{x^2-9}$

$g'(x) = \frac{1}{2}(x^2-9)^{-1/2} \cdot 2x$

$= \dfrac{x}{\sqrt{x^2-9}}$ ✓

7. $f(x) = \dfrac{x}{x+1}$

$y = \dfrac{x}{x+1}$

$y(x+1) = x$

$y = x - xy = x(1-y)$

$x = \dfrac{y}{1-y}$

So: $g(x) = \dfrac{x}{(1-x)}$

$f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2}$

$= \dfrac{1}{(x+1)^2}$

So $g'(x) = \dfrac{1}{\dfrac{1}{\left[\left(\frac{x}{1-x}\right)+1\right]^2}}$

$= \left(\frac{x}{1-x}+1\right)^2 = \boxed{\dfrac{1}{(1-x)^2}}$

8. $f(x) = 2 + x^{-1}$

$y = 2 + x^{-1}$
$y - 2 = x^{-1}$, so $x = \frac{1}{y-2}$.

So $g(x) = \frac{1}{x-2}$.

But also $f'(x) = -x^{-2}$.

So $g'(x) = \frac{1}{f'(g(x))} = \frac{1}{-\left(\frac{1}{x-2}\right)^{-2}} = \boxed{\frac{-1}{(x-2)^2}}$

10. $f(x) = \frac{x^3}{x^2+1}$

Want $g'(-\frac{1}{2}) = \frac{1}{f'(g(-\frac{1}{2}))}$

$f'(x) = \frac{(x^2+1)(3x^2) - x^3(2x)}{(x^2+1)^2} = \frac{x^4 + 3x^2}{(x^2+1)^2}$

If $g(-\frac{1}{2}) = x$, $f(x) = \frac{-1}{2}$ since f and g are inverses.

$\frac{x^3}{x^2+1} = \frac{-1}{2}$ when $x = -1$.

So $g(-\frac{1}{2}) = -1$, and $f'(g(-\frac{1}{2})) = f'(-1) = \frac{(-1)^4 + 3(-1)^2}{((-1)^2+1)^2}$

$= \frac{1+3}{2^2} = 1.$

Thus, $g'(-\frac{1}{2}) = \frac{1}{f'(g(-\frac{1}{2}))} = \frac{1}{1} = \boxed{1}$.

18. $f(x) = \frac{1}{1+x}$, $g(x) = \frac{1-x}{x}$

To show f and g are inverses, compute $f \circ g$ and $g \circ f$

$f \circ g(x) = \frac{1}{1 + \frac{1-x}{x}} = \frac{1}{\frac{x+1-x}{x}} = \frac{1}{\frac{1}{x}} = x$ ✓

$g \circ f(x) = \frac{1 - \frac{1}{1+x}}{\frac{1}{1+x}} = \frac{\cancel{1}+x\cancel{-1}}{\cancel{1+x}} \cdot \frac{\cancel{1+x}}{1} = x$ ✓

Compute $g'(x)$ directly: $g'(x) = \frac{x(-1) - (1-x)(1)}{x^2} = \frac{\cancel{-x} - 1 + \cancel{x}}{x^2} = \frac{-1}{x^2}$

Compute $\frac{1}{f'(g(x))}$: $f'(x) = \frac{d}{dx}(1+x)^{-1} = -(1+x)^{-2} = \frac{-1}{(1+x)^2}$.

So $f'(g(x)) = \frac{-1}{\left(1 + \frac{1-x}{x}\right)^2} = \frac{-1}{\left(\frac{x+1-x}{x}\right)^2} = -x^2$.

and $\frac{1}{f'(g(x))} = \frac{-1}{x^2}$ ✓

23. $y = \sin^{-1}(7x)$

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, so use the chain rule:

$$y' = \frac{1}{\sqrt{1-(7x)^2}} \cdot 7 = \boxed{\frac{7}{\sqrt{1-49x^2}}}$$

24. $y = \arctan\left(\frac{x}{3}\right)$

Know: $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

Use the chain rule: $y' = \frac{1}{1+\left(\frac{x}{3}\right)^2} \cdot \frac{1}{3} = \boxed{\frac{1}{3+\frac{x^2}{3}}}$

28. $y = e^{\cos^{-1}x}$

Know: $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

Use the chain rule. $y' = e^{\cos^{-1}x} \cdot \frac{-1}{\sqrt{1-x^2}} = \frac{-e^{-\cos^{-1}x}}{\sqrt{1-x^2}}$

32. $y = \tan^{-1}\left(\frac{1+t}{1-t}\right)$

$$y' = \frac{1}{1+\left(\frac{1+t}{1-t}\right)^2} \cdot \frac{d}{dt}\left(\frac{1+t}{1-t}\right) = \frac{1}{1+\left(\frac{1+t}{1-t}\right)^2} \cdot \left[\frac{(1-t)+(1+t)}{(1-t)^2}\right]$$

$$= \frac{1}{1+\left(\frac{1+t}{1-t}\right)^2} \cdot \frac{2}{(1-t)^2} = \frac{2}{(1-t)^2+(1+t)^2}$$

$$= \frac{2}{1-2t+t^2+1+2t+t^2} = \frac{1}{1+t^2}$$

Section 3.9

Preliminary Questions

1. The slope of the tangent line to $y = 4^x$ at $x=0$ is $\boxed{\ln 4}$

since $\frac{d}{dx}4^x = (\ln 4)\,4^x$ and $\left.\frac{d}{dx}4^x\right|_{x=0} = \ln 4.$

4. If $(\log_b x)' = \frac{1}{3x}$, what is b?

$(\log_b x)' = \frac{1}{(\ln b)x}$, so $\ln b = 3 \Rightarrow \boxed{b = e^3}$

Section 2.9

Exercises

25. $f(x) = 6^x, \quad x=2$

$f'(x) = (\ln 6)\, 6^x$

$f'(2) = 36\ln 6$ = slope of tangent line to $y=6^x$ when $x=2$.

The point on $y=6^x$ with $x=2$: $y = 6^2 = 36$, so point: $(2, 36)$

$\boxed{y-36 = 36\ln 6\,(x-2)}$

29. $f(x) = 5^{x^2-2x}, \quad x=1$

$f'(x) = (\ln 5)\, 5^{x^2-2x} \cdot (2x-2)$

$f'(1) = \ln 5 \cdot 5^{1-2}\,(2-2) = 0$, so the tangent line is horizontal.

When $x=1$, $y=f(1) = 5^{1-2} = \frac{1}{5}$

So the tangent line is $\boxed{y = \frac{1}{5}}$

32. $f(x) = \ln(x^2) \quad x=4.$

$f'(x) = \frac{1}{x^2} \cdot 2x$

$f'(4) = \frac{8}{16} = \frac{1}{2}.$

$f(4) = \ln 16.$

So the tangent line is $\boxed{y - \ln 16 = \frac{1}{2}(x-4)}$

35. $f(w) = \log_2 w, \quad w = \frac{1}{8}$

$f'(w) = \frac{1}{(\ln 2)\, w}$, so $f'(\frac{1}{8}) = \frac{1}{\frac{1}{8}\ln 2} = \frac{8}{\ln 2}$

$f(\frac{1}{8}) = \log_2 \frac{1}{8} = -3$

So the tangent line is $\boxed{y+3 = \frac{8}{\ln 2}\left(x - \frac{1}{8}\right)}$

78. Use: $\left(\ln(f(x))\right)' = \frac{f'(x)}{f(x)}$

$(\ln x)' = \frac{1}{x}$ and $(\ln 2x)' = \frac{2}{2x} = \frac{1}{x}$

The simple explanation is that $\ln 2x = \ln 2 + \ln x$, so $\ln(2x)$ is just $\ln x$ plus a constant. The graph of $\ln 2x$ is the same as the graph of $\ln x$, but shifted up.

80. $\log_{10} E = 4.8 + 1.5M$.

a) Show that when M increases by one, energy increases by a factor of 31.5

$E(M) = 10^{4.8+1.5M}$

$E(M+1) = 10^{4.8+1.5(M+1)}$

$$\frac{E(M+1)}{E(M)} = \frac{10^{4.8+1.5M+1.5}}{10^{4.8+1.5M}} = 10^{1.5} \approx 31.5$$

So when M increases by 1, E increases by a factor of about 31.5

b) $E(M) = 10^{4.8+1.5M}$

$$\frac{dE}{dM} = \ln 10 \left(10^{4.8+1.5M}\right) \cdot 1.5$$

$$= \boxed{1.5 \cdot \ln 10 \cdot \left(10^{4.8+1.5M}\right)}$$