Homework 5 (Due 10/11)
Solutions

Section 2.8

5. Show $\cos x = x$ has a solution in the interval $[0,1]$.

Consider $f(x) = x - \cos x$

f is continuous since x and $\cos x$ are continuous, and sums and diferences of continuous functions are continuous.

$f(0) = 0 - \cos 0 = 0 - 1 = -1$

$f(1) = 1 - \cos 1 > 0$ since $-1 \le \cos x \le 1$ for any x.

So by the IVT, $f(x) = x - \cos x = 0$ for some x in the interval $(0,1)$.

Thus, $x = \cos x$ has a solution in $[0,1]$.

6. $f(x) = x^3 + 2x + 1$

Find an interval of length $\frac{1}{2}$ containing a root of $f(x)$

$f(x)$ is continuous since it's a polynomial, so the IVT applies.

$f(-1) = (-1)^3 + 2(-1) + 1 = -1 - 2 + 1 = -2 < 0$

$f(0) = 0^3 + 2 \cdot 0 + 1 = 1$.

So by the IVT, $f(x)$ has a root in the interval $[-1, 0]$.

To get an interval of length $\frac{1}{2}$, cut our interval in half and check

$f(-\frac{1}{2}) = (-\frac{1}{2})^3 + 2(-\frac{1}{2}) + 1 = -\frac{1}{8} - 1 + 1 = -\frac{1}{8}$.

So using the IVT again, f has a root in the interval $[-\frac{1}{2}, 0]$.

12. Show $2^x = bx$ has a solution if $b > 2$.

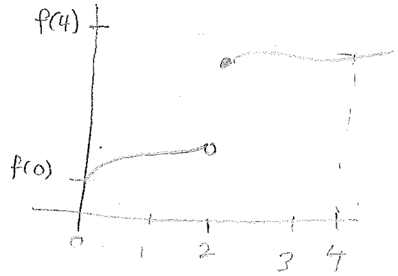
$2^x = bx$ has a solution exactly when $f(x) = 2^x - bx$ has a root.

$f(x)$ is continuous since 2^x and bx are continuous

$f(0) = 2^0 - b \cdot 0 = 1$ and $f(1) = 2^1 - b = 2 - b < 0$ when $b > 2$.

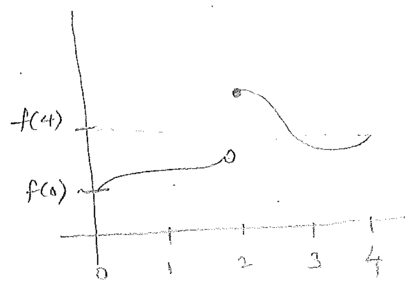
So the IVT tells us that $f(x)$ has a zero in the range $[0,1]$, and thus $2^x = bx$ has a solution in the same range

21. Jump discontinuity at $x=2$ and does not satisfy the conclusion of IVT.



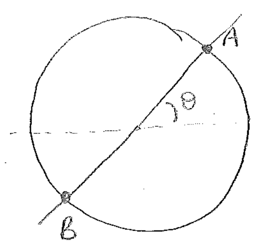
There are values between $f(0)$ and $f(4)$ that get skipped.

22. Jump discontinuity at $x=2$ and satisfies the conclusion of IVT.



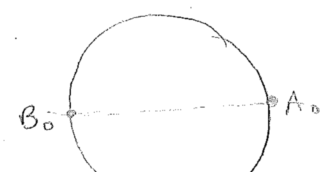
No values between $f(0)$ and $f(4)$ get skipped even though f is not continuous.

26. Let $f(\theta) = t(A) - t(B)$ where $t(A)$: temperature at point A
$t(B)$: temperature at point B.

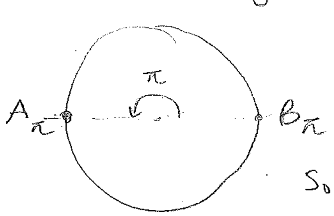


f is continuous because the temperature function is continuous (temperatures at nearby locations are close to each other).

Consider $f(0) = t(A_0) - t(B_0)$



If $f(0)=0$, then the temperatures at A_0 and B_0 are equal.
But maybe $f(0) \neq 0$. Then consider $f(\pi) = t(A_\pi) - t(B_\pi)$



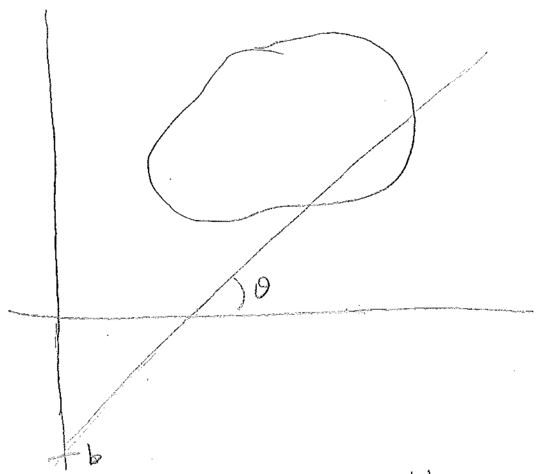
But $A_\pi = B_0$ and $B_\pi = A_0$, so

$f(\pi) = t(B_0) - t(A_0) = -f(0)$.

So $f(\pi)$ and $f(0)$ have opposite signs.

Thus, $f(c)=0$ for some c in the interval $[0, \pi]$.

29. For any angle θ, it is possible to cut a piece of ham in half with a linear cut of incline θ.

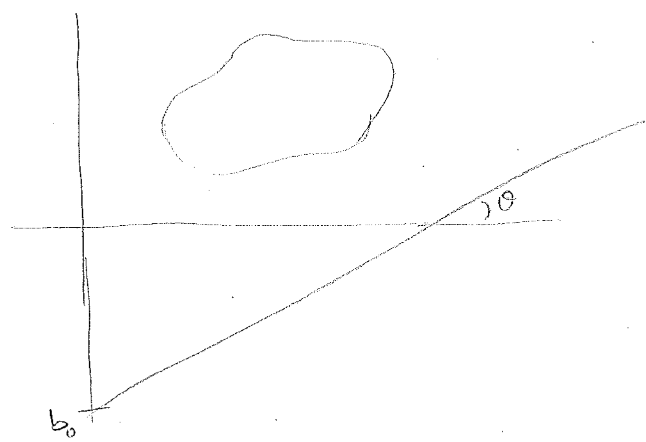


To each y-intercept b, we associate a line of incline θ, as shown. (If $\theta = \frac{\pi}{2}$ and the lines are all vertical, use x-intercepts instead).

Let $f(b) =$ (Amount of ham above/to the left of the line) − (Amount of ham below/to the right of the line).

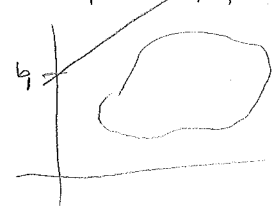
We want to show $f(b) = 0$ for some b, so the amount of ham on one side of the line is equal to the amount of ham on the other.

If b_0 is negative enough, the line misses the ham entirely and $f(b_0) = A$, where A = total amount of ham



All the ham on the left/above the line

If b_1 is positive enough, the line misses the ham entirely and $f(b_1) = -A$, where A = total amount of ham.



All the ham on the right/below the line.

Thus, by the intermediate value theorem, for some b between b_0 and b_1, $f(b) = 0$, and the corresponding line splits the ham into equal halves.

Section 3.1

24. $f(x) = \sqrt{x}$

$f(5+h) = \sqrt{5+h}$ and the difference quotient with $a=5$, $h=1$ is

$$\frac{f(a+h)-f(a)}{h} = \frac{f(6)-f(5)}{1} = \boxed{\sqrt{6}-\sqrt{5}}$$

28. $f(x) = 4-x^2$, $a=-1$.

$$f'(-1) = \lim_{x\to -1} \frac{f(x)-f(-1)}{x+1} = \lim_{x\to -1} \frac{4-x^2-(4-1)}{x+1}$$

$$= \lim_{x\to -1} \frac{4-x^2-3}{x+1} = \lim_{x\to -1} \frac{1-x^2}{x+1} = \lim_{x\to -1} \frac{-\cancel{(x+1)}(x-1)}{\cancel{x+1}}$$

$$= \lim_{x\to -1} -(x-1) = 2.$$

$f(-1) = 3$, so the equation of the tangent line is

$$\boxed{y-3 = 2(x+1)}$$

38. $f(t) = \sqrt{3t+5}$, $a=-1$.

$$f'(-1) = \lim_{x\to -1} \frac{f(x)-f(-1)}{x+1} = \lim_{x\to -1} \frac{\sqrt{3x+5}-\sqrt{2}}{x+1}$$

$$= \lim_{x\to -1} \frac{\sqrt{3x+5}-\sqrt{2}}{x+1} \cdot \frac{\sqrt{3x+5}+\sqrt{2}}{\sqrt{3x+5}+\sqrt{2}} = \lim_{x\to -1} \frac{3x+5-2}{(x+1)(\sqrt{3x+5}+\sqrt{2})}$$

$$= \lim_{x\to -1} \frac{3\cancel{(x+1)}}{\cancel{(x+1)}(\sqrt{3x+5}+\sqrt{2})} = \frac{3}{\sqrt{2}+\sqrt{2}} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

$f(-1) = \sqrt{2}$, so the equation of the line is

$$\boxed{y-\sqrt{2} = \frac{3\sqrt{2}}{4}(x+1)}$$

42. $f(x) = x^{-2}$, $a = -1$

$$f'(-1) = \lim_{x \to -1} \frac{x^{-2} - 1}{x+1} = \lim_{x \to -1} \frac{\frac{1}{x^2} - 1}{x+1} = \lim_{x \to -1} \frac{1 - x^2}{x^2(x+1)}$$

$$= \lim_{x \to -1} \frac{-\cancel{(x+1)}(x-1)}{x^2\cancel{(x+1)}} = \lim_{x \to -1} \frac{1-x}{x^2} = \frac{2}{1} = 2.$$

$f(-1) = 1$, so the equation of the tangent line is

$$\boxed{y - 1 = 2(x+1)}$$

43. $f(x) = \dfrac{1}{x^2+1}$, $a = 0$.

$$f'(0) = \lim_{x \to 0} \frac{\frac{1}{x^2+1} - 1}{x - 0} = \lim_{x \to 0} \frac{\frac{1}{x^2+1} - 1}{x}$$

$$= \lim_{x \to 0} \frac{1 - (x^2+1)}{x(x^2+1)} = \lim_{x \to 0} \frac{-x^{\cancel{2}}}{\cancel{x}(x^2+1)} = \lim_{x \to 0} \frac{-x}{(x^2+1)} = \frac{-0}{1}$$

$$= 0.$$

$f(0) = 1$, so the equation of the tangent line is

$$\boxed{y = 1}$$

45. By sketching the line tangent to the graph at $x = 4$, estimate that the slope is about $\frac{2.5}{3} \approx 0.83$.

The slope of the tangent line is 0 at $t = 10$ and $t \approx 11.5$

The slope is negative for t in the interval $(10, 11.5)$.

49. The derivative is positive where the function is increasing, namely, on the intervals $(1.0, 2.5)$ and $(3.5, 4.5)$.

56. $\lim_{h \to 0} \dfrac{5^h - 1}{h}$ $\quad f(x) = 5^x$ and $a = 0$. Then

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{5^h - 5^0}{h}$$

$$= \lim_{h \to 0} \frac{5^h - 1}{h}.$$

Section 3.2

Preliminary questions.

1. If $f'(x) = x^3$, the tangent line through $(2, f(2))$ has slope $f'(2) = 8$.

2. $(f-g)'(1) = (f'-g')(1) = f'(1) - g'(1) = 3-5 = -2$.

3. The power rule applies to

a) $f(x) = x^2$

c) $f(x) = x^e$

f) $f(x) = x^{-4/5}$ (it applies to $f(x) = x^{\{\text{number}\}}$)

Exercises

2. $f(x) = x^2 + 3x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) - (x^2+3x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h - \cancel{x^2} - \cancel{3x}}{h}$$

$$= \lim_{h \to 0} (2x + h + 3) = \boxed{2x+3}$$

6. $f(x) = x^{-1/2}$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^{-1/2} - x^{-1/2}}{h} = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h \cdot \sqrt{x(x+h)}} = \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x(x+h)}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{h\sqrt{x(x+h)}\,(\sqrt{x} + \sqrt{x+h})} = \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}\sqrt{x(x+h)}\,(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x(x+h)}\,(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{\sqrt{x^2}\,(\sqrt{x} + \sqrt{x})} = \frac{-1}{2x\sqrt{x}}$$

$$= \boxed{-\tfrac{1}{2} x^{-3/2}}$$