Homework 4 (Due 10/4)

Solutions

Section 2.6

Preliminary questions

1. $-x^4 \leq f(x) \leq x^2$.

$\lim_{x\to 0} f(x) = 0$ since $\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^2 = 0$.

There isn't enough information to determine $\lim_{x\to 1/2} f(x)$.

because $\lim_{x\to 1/2} -x^4 = -\frac{1}{16}$ and $\lim_{x\to 1/2} x^2 = 1/4$.

Therefore, $\lim_{x\to 1/2} f(x)$ could be anywhere between these.

Exercises

3. The squeeze theorem says $\lim_{x\to 7} f(x) = 6$.

Even though the inequality $f(x) \leq u(x)$ doesn't hold for all x, it holds for values of x near 7, so the Squeeze theorem still applies.

5. (a) $4x-5 \leq f(x) \leq x^2$

$\lim_{x\to 1} 4x-5 = -1$ and $\lim_{x\to 1} x^2 = 1$. Since these aren't equal, we <u>can't</u> determine $\lim_{x\to 1} f(x)$

(b) $2x-1 \leq f(x) \leq x^2$.

$\lim_{x\to 1} 2x-1 = 1 = \lim_{x\to 1} x^2$, so we <u>can</u> determine $\lim_{x\to 1} f(x) = 1$.

(c) $4x-x^2 \leq f(x) \leq x^2+2$.

$\lim_{x\to 1} 4x-x^2 = 3 = \lim_{x\to 1} x^2+2$, so we <u>can</u> determine $\lim_{x\to 1} f(x) = 3$.

8. $\lim_{x \to 0} x \sin \frac{1}{x^2}$

We know $-1 \le \sin \frac{1}{x^2} \le 1$ since the range of sin is $[-1, 1]$.

So for $x > 0$, $-x \le x \sin \frac{1}{x^2} \le x$.

For $x < 0$, $x \le x \sin \frac{1}{x^2} \le -x$ (multiplying inequalities by negative numbers flips the sign)

In other words,

$$-|x| \le x \sin \frac{1}{x^2} \le |x|$$

But $\lim_{x \to 0} -|x| = 0 = \lim_{x \to 0} |x|$. Therefore, $\boxed{\lim_{x \to 0} x \sin \frac{1}{x^2} = 0}$ also.

22. $\lim_{t \to \frac{\pi}{2}} \frac{1 - \cos t}{t} = \frac{\lim_{t \to \frac{\pi}{2}} (1 - \cos t)}{\lim_{t \to \frac{\pi}{2}} t}$ using basic limit laws

$= \frac{1 - \cos \frac{\pi}{2}}{\frac{\pi}{2}}$ using continuity of cos.

$= \frac{1 - 0}{\frac{\pi}{2}} = \frac{1}{\frac{\pi}{2}} = \boxed{\frac{2}{\pi}}$

Section 2.7

Preliminary questions

1. $\lim_{x \to \infty} f(x) = L$ and $\lim_{x \to L} g(x) = \infty$.

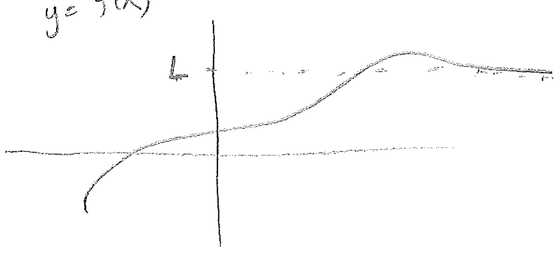
a) $x = L$ is a vertical asymptote of g TRUE

b) $y = L$ is a horizontal asymptote of g. FALSE

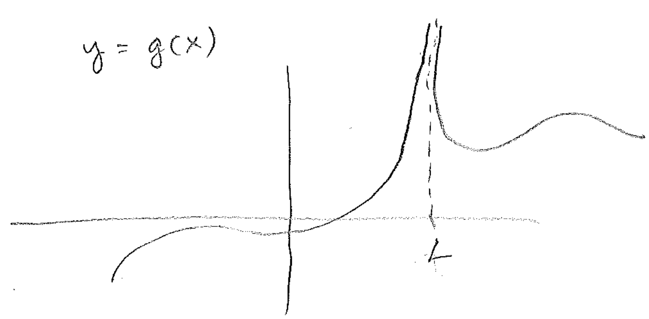
c) $x = L$ is a vertical asymptote of f. FALSE

d) $y = L$ is a horizontal asymptote of f. TRUE

$y = f(x)$



$y = g(x)$



6. $\lim_{x\to\infty} \sin\frac{1}{x}$ exists because as $x \to \infty$, $\frac{1}{x}$ goes to a number, namely 0.

So $\lim_{x\to\infty} \sin\frac{1}{x} = \sin 0 = 0.$

On the other hand, $\lim_{x\to 0} \sin\frac{1}{x}$ doesn't exist because as $x \to 0$, $\frac{1}{x}$ goes to $\pm\infty$. In other words, as x approaches 0, the thing we're plugging in to sin grows without bound. This just means we will wrap around the unit circle more and more times, but not necessarily settling toward a certain angle. $\sin\frac{1}{x}$ will just oscillate wildly as $x \to 0$.

Exercises

7. $$\lim_{x\to\infty} \frac{x}{x+9} = \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{x+9}{x}} = \lim_{x\to\infty} \frac{1}{1+\frac{9}{x}} = \frac{1}{1+0} = \boxed{1}$$

18. $$\lim_{x\to\infty} \frac{8x^3-x^2}{7+11x-4x^4} = \lim_{x\to\infty} \frac{\frac{1}{x^4}(8x^3-x^2)}{\frac{1}{x^4}(7+11x-4x^4)} = \lim_{x\to\infty} \frac{\frac{8}{x}-\frac{1}{x^2}}{\frac{7}{x^4}+\frac{11}{x^3}-4}$$

$$= \frac{0-0}{0+0-4} = \frac{0}{-4} = 0$$

$$\lim_{x\to-\infty} \frac{8x^3-x^2}{7+11x-4x^4} = \lim_{x\to-\infty} \frac{\frac{8}{x}-\frac{1}{x^2}}{\frac{7}{x^4}+\frac{11}{x^3}-4} = \frac{0}{-4} = 0.$$

$f(x) = \frac{8x^3-x^2}{7+11x-4x^4}$ has horizontal asymptotes at $\boxed{y=0}$

20. $$\lim_{x\to\infty} \frac{\sqrt{36x^4+7}}{9x^2+4} = \lim_{x\to\infty} \frac{\frac{1}{x^2}\sqrt{36x^4+7}}{\frac{1}{x^2}(9x^2+4)} = \lim_{x\to\infty} \frac{\sqrt{36+\frac{7}{x^4}}}{9+\frac{4}{x^2}}$$

$$= \frac{\sqrt{36}}{9} = \frac{6}{9} = \frac{2}{3}.$$

$$\lim_{x\to-\infty} \frac{\sqrt{36x^4+7}}{9x^2+4} = \lim_{x\to-\infty} \frac{\sqrt{36+7/x^4}}{9+4/x^2} = \frac{\sqrt{36}}{9} = \frac{6}{9} = \frac{2}{3}.$$

So $f(x)$ has horizontal asymptotes at $\boxed{y=\frac{2}{3}}$

33. $R(s) = \dfrac{As}{K+s}$ (A, K constants)

a) $\lim\limits_{s\to\infty} R(s) = \lim\limits_{s\to\infty} \dfrac{As}{K+s} = \lim\limits_{s\to\infty} \dfrac{A}{\frac{K}{s}+1} = A.$

So the limiting reaction rate is A.

b) When $s = K$, $R(s) = \dfrac{AK}{K+K} = \dfrac{AK}{2K} = \dfrac{A}{2}$, so the reaction rate is half the limiting rate (A).

c) $K = 1.25$, $A = 0.1$

Want $R(s) = 0.75A = 0.075$

$$\frac{0.1s}{1.25+s} = 0.075$$

$$0.1s = 0.09375 + 0.075s$$

$$0.025s = 0.09375$$

$$s = \frac{0.09375}{0.025} = \boxed{3.75}$$

34. $T(t) = 283 + 3(1-e^{-0.03t})$

a) $\lim\limits_{t\to\infty} T(t) = \lim\limits_{t\to\infty} 283 + 3(1-e^{-0.03t})$

$= 283 + 3 = \boxed{286}$ ($\lim\limits_{t\to\infty} e^{-0.03t} = 0$ since as t gets larger, $e^{-0.03t} = \frac{1}{e^{0.03t}}$ gets very close to zero).

Want:

b) $T(t) = 285.5$

$$283 + 3(1-e^{-0.03t}) = 285.5$$

$$3(1-e^{-0.03t}) = 2.5$$

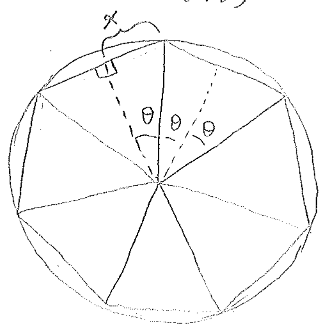
$$e^{-0.03t} = 1 - \frac{2.5}{3}$$

$$\ln\left(1-\frac{2.5}{3}\right) = -0.03t$$

$$t = -\frac{\ln\left(1-\frac{2.5}{3}\right)}{0.03} = \boxed{59.7}$$

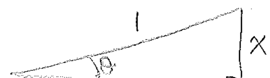
43. a) As n increases, the polygon inscribed in the circle becomes more and more like the circle, so the value of $P(n)$ approaches the circumference of the circle as n increases. Thus $\lim_{n\to\infty} P(n) = 2\pi$ = circumference of the unit circle

b) $P(n) = 2n \sin\left(\frac{\pi}{n}\right)$:



Each angle θ is $\frac{2\pi}{2n} = \frac{\pi}{n}$.

Looking at each of the triangles, we have:



The hypotenuse is 1 since the hypotenuse is a radius of the circle, and the circle is a unit circle.

Thus, $x = \sin\theta$.

Then $P(n) = 2n\sin\theta = 2n \sin\frac{\pi}{n}$.

c) $$\lim_{n\to\infty} \frac{n}{\pi}\sin\frac{\pi}{n} = \lim_{n\to\infty} \frac{2n\sin\frac{\pi}{n}}{2\pi} = \frac{\lim_{n\to\infty} P(n)}{\lim_{n\to\infty} 2\pi} = \frac{2\pi}{2\pi} = 1.$$

d) Let $\theta = \frac{\pi}{n}$. Then $\frac{n}{\pi} = \frac{1}{\theta}$, and as $n \to \infty$, $\theta \to 0$.

So from c, we have

$$\lim_{n\to\infty} \frac{n}{\pi}\sin\frac{\pi}{n} = \lim_{\theta\to 0} \frac{1}{\theta}\sin\theta = 1.$$

44. $$\lim_{c\to\infty} h(c) = \lim_{c\to\infty} c\sqrt{\frac{c^2}{g^2}+\frac{1}{4}} - \frac{c^2}{g} = \lim_{c\to\infty} c^2\left(\sqrt{\frac{1}{g^2}+\frac{1}{4c^2}} - \frac{1}{g}\right)$$

$$= \lim_{c\to\infty} c^2\left(\sqrt{\frac{1}{g^2}+\frac{1}{4c^2}} - \frac{1}{g}\right)\frac{\left(\sqrt{\frac{1}{g^2}+\frac{1}{4c^2}} + \frac{1}{g}\right)}{\left(\sqrt{\frac{1}{g^2}+\frac{1}{4c^2}} + \frac{1}{g}\right)}$$

$$= \lim_{c\to\infty} \frac{c^2\left(\cancel{\frac{1}{g^2}} + \frac{1}{4c^2} - \cancel{\frac{1}{g^2}}\right)}{\sqrt{\frac{1}{g^2}+\frac{1}{4c^2}} + \frac{1}{g}} = \lim_{c\to\infty} \frac{c^2\left(\frac{1}{4c^2}\right)}{\sqrt{\frac{1}{g^2}+\frac{1}{4c^2}} + \frac{1}{g}}$$

$$= \lim_{c\to\infty} \frac{\frac{1}{4}}{\sqrt{\frac{1}{g^2}+\frac{1}{4c^2}} + \frac{1}{g}} = \frac{\frac{1}{4}}{\sqrt{\frac{1}{g^2}+0} + \frac{1}{g}} = \frac{\frac{1}{4}}{\frac{1}{g}+\frac{1}{g}} = \frac{\frac{1}{4}}{\frac{2}{g}} = \boxed{\frac{g}{8}}$$ ✓