Homework 2 (Due 9/20)

Solutions

Section 1.5

Preliminary questions.

3. f maps teenagers in the US to their last names. f^{-1} does not exist because f is not one-to-one. For example, there are many teenagers with the last name Smith, so f^{-1}(Smith) is impossible to define precisely.

4. The function f from towns to times is one-to-one. $f^{-1}(6{:}27)=$ Hamilton Township.

Exercises

4. $f(x)=\dfrac{x-2}{x+3}$

$y=\dfrac{x-2}{x+3}$, so $y(x+3)=x-2$

$xy+3y=x-2$

$3y+2=x-xy=x(1-y)$

So $x=\dfrac{3y+2}{1-y}$

$f^{-1}(y)=\dfrac{3y+2}{1-y}$

Check: $f\circ f^{-1}(y)=f\left(\dfrac{3y+2}{1-y}\right)=\dfrac{\frac{3y+2}{1-y}-2}{\frac{3y+2}{1-y}+3}=\dfrac{3y+\cancel{2}-\cancel{2}+2y}{\cancel{3y}+2+3-\cancel{3y}}=\dfrac{5y}{5}=y$ ✓

$f^{-1}\circ f(x)=f^{-1}\left(\dfrac{x-2}{x+3}\right)=\dfrac{3\frac{x-2}{x+3}+2}{1-\frac{x-2}{x+3}}=\dfrac{3x-\cancel{6}+2x+\cancel{6}}{\cancel{x}+3-\cancel{x}+2}=\dfrac{5x}{5}=x$ ✓

f is invertible because we've produced an inverse $\boxed{f^{-1}(y)=\dfrac{3y+2}{1-y}}$

a) The domain of f is real numbers x with $x\neq -3$. The range of f^{-1} is the same.

b) The domain of f^{-1} is real numbers y with $y\neq 1$. The range of f is the same.

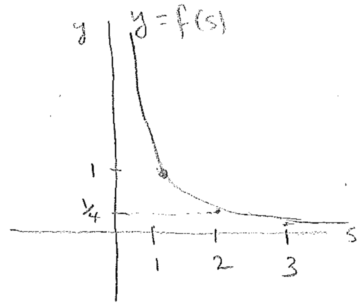
12. $f(s) = \frac{1}{s^2}$

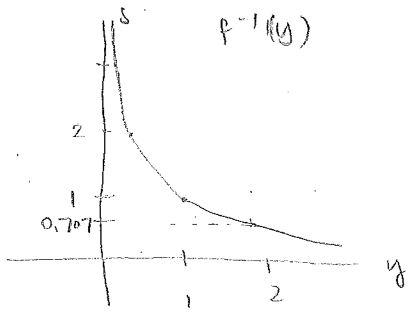
This function one-to-one on $(0, \infty)$.

$y = \frac{1}{s^2}$, so $s^2 = \frac{1}{y}$

$s = \frac{1}{\sqrt{y}}$

$$\boxed{f^{-1}(y) = \frac{1}{\sqrt{y}}}$$





20. $f(x) = (x^2+1)^{-1}$ is one-to-one on $(-\infty, 0]$ because f is increasing on this range. (x^2+1 gets smaller as x gets towards zero, and $\frac{1}{x^2+1}$ is increasing therefore). Alternatively, one could graph this function on the domain $(-\infty, 0]$ and see that the graph passes the horizontal line test. Yet another way to see that f is one-to-one: if a and b are both in the range $(-\infty, 0]$, with

$$\frac{1}{a^2+1} = \frac{1}{b^2+1} \text{, then } a^2 = b^2 \text{, so } a = \pm b.$$

But since both a, b are negative, $a = b$.

To find the inverse:

$$y = \frac{1}{x^2+1}$$

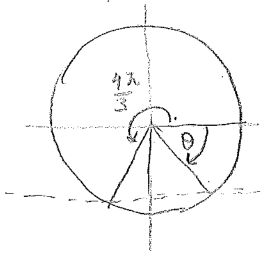
$$x^2+1 = \frac{1}{y}$$

$$x = -\sqrt{\frac{1}{y} - 1}$$

(Choose the negative square root since the range of f^{-1} should be x in $(-\infty, 0]$)

$$\boxed{f^{-1}(y) = -\sqrt{\frac{1}{y} - 1}}$$

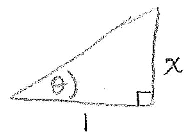
30. $\sin^{-1}\left(\sin\frac{4\pi}{3}\right)$



$\theta = \sin^{-1}\left(\sin\frac{4\pi}{3}\right) = \boxed{-\frac{\pi}{3}}$

$\left(\sin\frac{4\pi}{3} = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\right)$

40. $\cos(\tan^{-1} x)$



$\tan^{-1} x = \theta$

$\cos\theta = \frac{1}{\text{hypotenuse}}$, and $\text{hyp.}^2 = 1^2 + x^2$

So $\boxed{\cos\theta = \frac{1}{\sqrt{1+x^2}}}$

Section 1.6

1. a) $7^0 = \boxed{1}$

b) $10^2(2^{-2}+5^{-2}) = \frac{10^2}{4} + \frac{10^2}{25} = 25+4 = \boxed{29}$

c) $\frac{(4^3)^5}{(4^5)^3} = \frac{4^{15}}{4^{15}} = \boxed{1}$

d) $27^{4/3} = (27^{1/3})^4 = 3^4 = \boxed{81}$

e) $8^{-1/3} \cdot 8^{5/3} = 8^{4/3} = (8^{1/3})^4 = 2^4 = \boxed{16}$

f) $3 \cdot 4^{1/4} - 12 \cdot 2^{-3/2} = 3 \cdot (2^2)^{1/4} - 3 \cdot 2^2 \cdot 2^{-3/2}$

$= 3 \cdot 2^{1/2} - 3 \cdot 2^{2-3/2}$

$= 3 \cdot 2^{1/2} - 3 \cdot 2^{1/2} = \boxed{0}$

6. $(\sqrt{5})^x = 125$

$(5^{1/2})^x = 5^3$

$\frac{x}{2} = 3$, so $\boxed{x = 6}$

19. $\log_8 2 + \log_4 2 = \frac{1}{3} + \frac{1}{2} = \boxed{\frac{5}{6}}$

$(\log_8 2 = \frac{1}{3}$ since $8^{1/3} = 2)$.

20. $\log_{25} 30 + \log_{25} 5/6 = \log_{25}(30 \cdot 5/6) = \log_{25}(25) = \boxed{1}$

26. $8^{3\log_8 2} = \left(8^{\log_8 2}\right)^3 = 2^3 = \boxed{8}$

34. $\log_3 y + 3\log_3 y^2 = 14$

$\log_3 y + 3 \cdot 2 \cdot \log_3 y = 14$

$= \log_3 y + 6\log_3 y = 7\log_3 y = 14$

So $\log_3 y = 2$ and $\boxed{y = 3^2 = 9}$

42. Given $\log_{10} E = 4.8 + 1.5M$

a) $E(M) = 10^{4.8+1.5M}$ (using the definition of logs).

b) $E(M+1) = 10^{4.8+1.5(M+1)}$

$= 10^{4.8+1.5M+1.5}$

This is $10^{1.5}$ times $E(M)$ $\left(\frac{E(M+1)}{E(M)} = \frac{10^{4.8+1.5M+1.5}}{10^{4.8+1.5M}} = 10^{1.5} \right)$

or approximately 31.6.

So if the magnitude increases by one, the energy gets multiplied by a factor of 31.6.

Section 2.1

8. $s(t) = t^3 + t$

Average velocity over $[1,4]$ is $\frac{s(4) - s(1)}{4 - 1} = \frac{68 - 2}{3} = \frac{66}{3} = \boxed{22}$

Estimate instantaneous velocity at 1 using small intervals around 1.

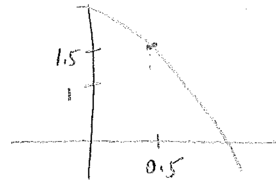
$[1, 1.1]$: $\frac{s(1.1) - s(1)}{0.1} = 4.31$

$[1, 1.01]$: $\frac{s(1.01) - s(1)}{0.01} = 4.0301$.

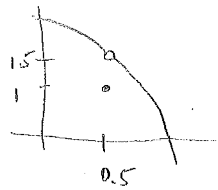
Instantaneous velocity at 1 is about 4.

Section 2.2

5. $\lim_{x \to 0.5} f(x) = 1.5$



6. $\lim_{x \to 0.5} g(x) = 1.5$



24. $\lim_{x \to 0} \frac{\sin x}{x^2} = ?$

$\lim_{x \to 0^-}$

x	$f(x)$
-0.1	$-9.983\ldots$
-0.01	$-99.9983\ldots$
-0.001	$-999.99983\ldots$

$f(x) \to -\infty$.

$\lim_{x \to 0^+}$

x	$f(x)$
0.1	$9.983\ldots$
0.01	$99.9983\ldots$
0.001	$999.99983\ldots$

$f(x) \to \infty$

$\lim_{x \to 0} \frac{\sin x}{x^2}$ does not exist, but $\lim_{x \to 0^-} \frac{\sin x}{x^2} = -\infty$

$\lim_{x \to 0^+} \frac{\sin x}{x^2} = \infty$.

27. $\lim_{x\to 3^+} \frac{x-4}{x^2-9}$

x	$f(x)$
3.1	-1.4754...
3.01	-16.47...
3.001	-166.47...
3.0001	-1666.472...

So $\lim_{x\to 3^+} \frac{x-4}{x^2-9} = -\infty$.

38. $\lim_{x\to 1^-} f(x) = 3$

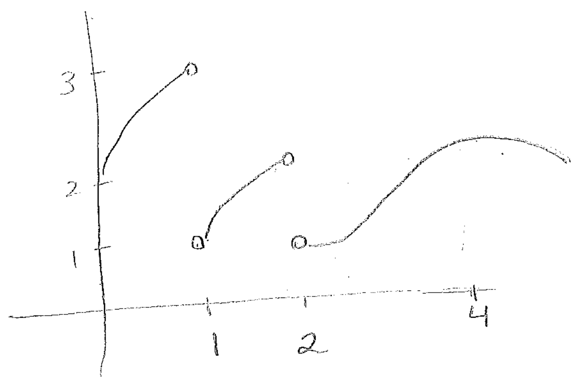
$\lim_{x\to 1^+} f(x) = 1$

$\lim_{x\to 2^-} f(x) = 2$

$\lim_{x\to 2^+} f(x) = 1$

$\lim_{x\to 4^-} f(x) = 2$

$\lim_{x\to 4^+} f(x) = 2$.



$\lim_{x\to 1} f(x)$ does not exist

$\lim_{x\to 2} f(x)$ does not exist

$\lim_{x\to 4} f(x) = 2$.

54. For A, the limit as $x\to 0$ does not exist (the function is still oscillating wildly near 0).

For B, the limit as $x\to 0$ is 0.