Homework 12 (Due 12/9)
Solutions

Section 4.3

24. $y = 5x^2 + 6x - 4$

Critical points: $y' = 10x + 6 = 0$

$x = \frac{-3}{5}$

For the interval $(-\infty, \frac{-3}{5})$, $10x+6 < 0$ and y is decreasing

(e.g test $y'(-1) = -10+6 = -4$)

For the interval $(\frac{-3}{5}, \infty)$, $10x+6 > 0$ and y is increasing

(e.g. test $y'(1) = 10+6 = 16 > 0$)

$y = 5x^2+6x-4$ is decreasing on $(-\infty, \frac{-3}{5})$ and increasing on $(-\frac{3}{5}, \infty)$.

The only critical point is $x = \frac{-3}{5}$, where there is a local minimum.

51. $y = x - \ln x$ $(x > 0)$

Critical points: $y' = 1 - \frac{1}{x}$

$y'(0)$ does n't exist, but isn't in the domain of $y = x - \ln x$.

So the only critical points are where

$y' = 1 - \frac{1}{x} = 0$, i.e. $x = 1$.

For $(0,1)$: Test $y'(\frac{1}{2})$: $1 - \frac{1}{\frac{1}{2}} = 1 - 2 = -1 < 0$

For $(1, \infty)$ Test $y'(2) = 1 - \frac{1}{2} = \frac{1}{2} > 0$.

So $y = x - \ln x$ is decreasing on $(0,1)$ and increasing on $(1, \infty)$.

The only critical point is $x = 1$, and it is the location of a local minimum.

53. Find the minimum value of $f(x) = x^x$ for $x>0$

Find critical points: $f'(x) = ?$

Use logarithmic differentiation.

$$\ln f(x) = \ln x^x = x \ln x$$

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)} = x \cdot \frac{1}{x} + \ln x = 1 + \ln x.$$

So $f'(x) = x^x (1 + \ln x)$

For $x>0$, $f'(x)$ always exists. $f'(x) = 0$ when

$x^x = 0$ or $1 + \ln x = 0$.

But if $x>0$, $x^x > 0$. So consider $1+\ln x = 0$.

$$1 + \ln x = 0 \Rightarrow \ln x = -1$$

$$x = e^{-1} = \frac{1}{e}.$$

Check: on $(0, \frac{1}{e})$, $f'(x) < 0$ (e.g. test $\frac{1}{e^2}$:

$$f'(\tfrac{1}{e^2}) = \left(\frac{1}{e^2}\right)^{\frac{1}{e^2}} \left(1 + \ln\left(\frac{1}{e^2}\right)\right)$$

$$= \left(\frac{1}{e^2}\right)^{\frac{1}{e^2}} (1 + -2) = -\left(\frac{1}{e^2}\right)^{\frac{1}{e^2}} < 0)$$

on $(\frac{1}{e}, \infty)$, $f'(x) > 0$.

e.g. Test $f'(e) = e^e (1 + \ln e)$

$$= e^e (1+1) = 2e^e > 0)$$

So $f(x)$ is minimized at $x = \frac{1}{e}$ and the minimum value is $\left(\frac{1}{e}\right)^{\frac{1}{e}}$

Section 4.4

2. (a) ii

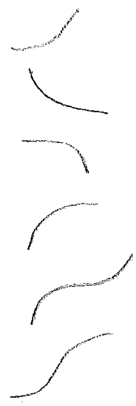
(b) iv

(c) i

(d) iii

(e) vi

(f) v



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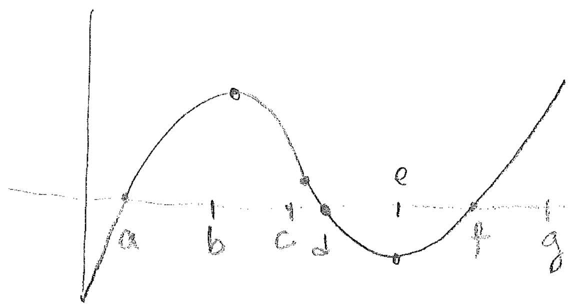


20. Point of inflection of f occurs at $x = c$.

$f(x)$ is concave down on $(0, c)$

21. If the graph is the graph of $f'(x)$:

Points of inflection of f occur at $x = b$ and $x = e$.

f is concave down when f' is decreasing, i.e. on (b, e)

22. If the graph shown is the graph of $f''(x)$:

Points of inflection of f occur at $x = a$ and $x = d$ and $x = f$.

f is concave down when $f'' < 0$, i.e. on $(0, a)$ and (d, f).

40. $f(x) = x^2(x-4)$

Find critical points:

$f(x) = x^3 - 4x^2.$

$f'(x) = 3x^2 - 8x = x(3x-8) = 0$ when $x = 0$ or when $x = \frac{8}{3}.$

Look at $f''(x) = 6x - 8 = 0$ when $x = \frac{8}{6} = \frac{4}{3}.$

$f'(x)$:

$(-\infty, 0)$	$(0, \frac{8}{3})$	$(\frac{8}{3}, \infty)$
$f'(-1) = 11$	$f'(1) = -5$	$f'(3) = 3$
+	−	+

$f''(x)$:

$(-\infty, \frac{4}{3})$	$(\frac{4}{3}, \infty)$
$f''(0) = -8$	$f''(2) = 4$
−	+

f is concave up on $(\frac{4}{3}, \infty)$

concave down on $(-\infty, \frac{4}{3})$

f has a point of inflection at $x = \frac{4}{3}$

The critical points of f are $x = 0$ and $x = \frac{8}{3}.$

At $x = 0$, f has a local maximum with value 0.

At $x = \frac{8}{3}$, f reaches a local minimum of 0.

52. $y = \ln(x^2+2x+5)$

$$y' = \frac{2x+2}{x^2+2x+5}$$

$$y'' = 2 \cdot \frac{1}{x^2+2x+5} + \frac{-1(2x+2)}{(x^2+2x+5)^2}(2x+2)$$

$$= \frac{2(x^2+2x+5) - (4x^2+8x+4)}{(x^2+2x+5)^2} = \frac{-2x^2-4x+6}{(x^2+2x+5)^2}$$

$y' = 0$ when $2x+2=0$, i.e. when $x = -1$.

(y' always exists since $x^2+2x+5 \neq 0$, so $x=-1$ is the only critical point).

$y''=0$ when $-2x^2-4x+6=0$, i.e when $x^2+2x-3=0$

$(x-1)(x+3)=0$

$y''=0$ when $x=1$ and when $x=-3$.

y':	$(-\infty, -1)$	$(-1, \infty)$
	$y'(-2) = -4$	$y'(0) = 2$
	$-$	$+$
	$\searrow$	$\nearrow$

y''	$(-\infty, -3)$	$(-3, 1)$	$(1, \infty)$
	$y''(-4) = \frac{-2}{13^2} < 0$	$y''(0) = \frac{6}{25}$	$y''(2) = \frac{-10}{13^2} < 0$
	$-$	$+$	$-$
	$\frown$	$\smile$	$\frown$

f is concave up on $(-3,1)$

concave down on $(-\infty,-3)$ and $(1,\infty)$.

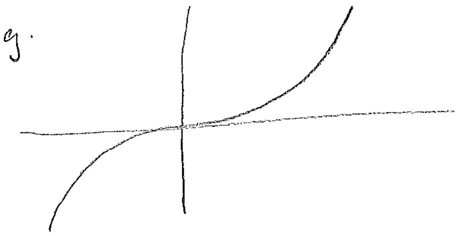
f has points of inflection at $x=-3$, $x=1$.

f has a critical point at $x=-1$, where f achieves a local minimum value of $\ln 8$

55. i) $f'(x) > 0$ for all x

ii) $f''(x) < 0$ for $x < 0$, $f''(x) > 0$ for $x > 0$.

e.g.



57. a) $R(t)$ = # infected individuals at time t.

$R(t)$ is concave up at the beginning of the epidemic and concave down near the end.

b) On the day that $R(t)$ has a point of inflection, the rate at which people are infected starts slowing down.

61. $g(u) = \dfrac{f(u) - f(0)}{f(1) - f(0)}$, $f(u) = (1 + e^{b(a-u)})^{-1}$

a) $f'(u) = -(1+e^{b(a-u)})^{-2} \cdot e^{b(a-u)} \cdot (-b)$

$$= \frac{be^{b(a-u)}}{(1+e^{b(a-u)})^2}.$$

The denominator is positive because it is a square.
The numerator is positive since $e^x > 0$ and $b > 0$.
So $f'(u) > 0$.

For $g(u)$: $g'(u) = \dfrac{f'(u)}{f(1) - f(0)}$ since $f(1)$, $f(0)$ are constants.

and $g'(u) > 0$ since $f'(u) > 0$ and $f(1) > f(0)$.

Also $g(0) = \dfrac{f(0) - f(0)}{f(1) - f(0)} = 0$

and $g(1) = \dfrac{f(1) - f(0)}{f(1) - f(0)} = 1$

So g is increasing on $[0, 1]$ and increases from 0 to 1.

61 b. $g(u)$ has a point of inflection when $g''(u)=0$ and g'' changes sign.

$$g''(u) = \frac{f''(u)}{f(1)-f(0)} = 0 \quad \text{when } f''(u)=0.$$

$$f''(u) = \frac{d}{du}\,\frac{b\,e^{b(a-u)}}{(1+e^{b(a-u)})^2}$$

$$= \frac{(1+e^{b(a-u)})^2\, b\,e^{b(a-u)}\cdot(-b) - b e^{b(a-u)}(2)(1+e^{b(a-u)})e^{b(a-u)}(-b)}{(1+e^{b(a-u)})^4}$$

$$= \frac{-b^2 e^{b(a-u)}(1+e^{b(a-u)}) + 2b^2(e^{b(a-u)})^2}{(1+e^{b(a-u)})^3}$$

So $f''(u)=0$ when $-b^2 e^{b(a-u)}(1+e^{b(a-u)}) + 2b^2(e^{b(a-u)})^2 = 0$,

i.e. when $-(1+e^{b(a-u)}) + 2e^{b(a-u)} = 0$

So $-1+e^{b(a-u)} = 0$

i.e. $e^{b(a-u)} = 1$

So $b(a-u)=0$ and $\underline{u=a}$.

Simplify $f''(u)$:
$$f''(u) = \frac{b^2 e^{b(a-u)}(-1-e^{b(a-u)}+2e^{b(a-u)})}{(1+e^{b(a-u)})^3}$$
$$= \frac{b^2 e^{b(a-u)}(e^{b(a-u)}-1)}{(1+e^{b(a-u)})^3}.$$

On $(0,a)$, $g''(u) = \dfrac{f''(u)}{f(1)-f(0)} > 0$ since $e^{b(a-u)} = e^{(\text{positive power})} > 1$

and thus $e^{b(a-u)}-1>0$.

On (a,∞), $g''(u) = \dfrac{f''(u)}{f(1)-f(0)} < 0$ since $e^{b(a-u)} = e^{\text{negative power}} < 1$

and thus $e^{b(a-u)}-1<0$.

So g'' changes sign at a and g has a point of inflection at $u=a$.

Section 4.5

Preliminary questions

1. You can't apply L'Hospital's rule to $\lim_{x\to 0} \frac{x^2-2x}{3x-2}$ since

$\lim_{x\to 0} x^2-2x = 0$ but $\lim_{x\to 0} 3x-2 = -2$.

So if we "plug in", we get $\frac{0}{-2}$, which is not an indeterminate form. L'Hospital only applies when we "plug in" and get $\frac{0}{0}$ or $\frac{\infty}{\infty}$

2. If $f(x)$ and $g(x)$ both approach ∞ as $x \to \infty$, L'Hospital's rule does not apply to $\lim_{x\to\infty} f(x)g(x)$.

In this case, $\lim_{x\to\infty} f(x)g(x) = \infty$.

Exercises.

18. $$\lim_{x\to 4} \frac{1}{\sqrt{x}-2} - \frac{4}{x-4} = \lim_{x\to 4} \frac{(x-4) - 4(\sqrt{x}-2)}{(\sqrt{x}-2)(x-4)}$$

$$= \lim_{x\to 4} \frac{x - 4\sqrt{x} + 4}{x^{3/2} - 2x - 4x^{1/2} + 8}$$ Plugging in gives $\frac{0}{0}$, so use L'Hospital

$$= \lim_{x\to 4} \frac{1 - 2x^{-1/2}}{\frac{3}{2}x^{1/2} - 2 - 2x^{-1/2}}$$ Plugging in gives $\frac{0}{0}$, so use L'Hospital again

$$= \lim_{x\to 4} \frac{x^{-3/2}}{\frac{3}{4}x^{-1/2} + x^{-3/2}} = \frac{4^{-3/2}}{\frac{3}{4}4^{-1/2} + 4^{-3/2}} = \frac{\frac{1}{8}}{\frac{3}{4}\cdot\frac{1}{2} + \frac{1}{8}}$$

$$= \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{8}\cdot\frac{8}{4} = \boxed{\frac{1}{4}}$$

20. $\lim_{x\to\infty} \frac{x^{2/3}+3x}{x^{5/3}-x}$. "Plugging in" gives $\frac{\infty}{\infty}$ which is indeterminate.

Option 1: Algebraic manipulations

$$\lim_{x\to\infty} \frac{x^{2/3}+3x}{x^{5/3}-x} = \lim_{x\to\infty} \frac{\frac{1}{x^{5/3}}\left(x^{2/3}+3x\right)}{\frac{1}{x^{5/3}}\left(x^{5/3}-x\right)}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{3}{x^{2/3}}}{1-\frac{1}{x^{2/3}}} = \frac{0+0}{1-0} = \frac{0}{1} = 0$$

Option 2: L'Hospital's Rule

$$\lim_{x\to\infty} \frac{x^{2/3}+3x}{x^{5/3}-x} = \lim_{x\to\infty} \frac{\frac{2}{3}x^{-1/3}+3}{\frac{5}{3}x^{2/3}-1}$$

Now "plugging in" gives $\frac{3}{\text{very large number}} \to 0$

26. $\lim_{x\to\frac{\pi}{2}} \frac{\tan 4x}{\tan 5x}$ Plugging in gives $\frac{\tan 2\pi}{\tan\frac{5\pi}{2}} = \frac{0}{\text{"}\infty\text{"}}$, so the limit is $\boxed{0}$ $\left(\frac{\text{very small number}}{\text{very large number}} \to 0\right)$

44. $\lim_{x\to\infty} e^{-x}(x^3-x^2+9)$ "Plugging in" gives $0\cdot\infty$, which is indeterminate

Rewrite this as $\lim_{x\to\infty} \frac{x^3-x^2+9}{e^x}$

Now "plugging in" gives $\frac{\infty}{\infty}$, which is still indeterminate, but we can apply L'Hospital.

$$\lim_{x\to\infty} \frac{x^3-x^2+9}{e^x} = \lim_{x\to\infty} \frac{3x^2-2x}{e^x} = \lim_{x\to\infty} \frac{6x-2}{e^x}$$

$$= \lim_{x\to\infty} \frac{6}{e^x} = \boxed{0}$$

46. $\lim_{x\to\infty} x^{1/x^2}$ "Plugging in" gives ∞^0 which is indeterminate.

Use ln to turn the exponential into a product.

$$\ln x^{1/x^2} = \frac{1}{x^2}\cdot \ln x = \frac{\ln x}{x^2}.$$

Now: $\lim_{x\to\infty} \frac{\ln x}{x^2}$ "Plugging in" gives $\frac{\infty}{\infty}$, which is indeterminate, but apply L'Hospital's rule.

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$$

So $\lim_{x\to\infty} x^{1/x^2} = \lim_{x\to\infty} e^{\ln x^{1/x^2}} = e^0 = \boxed{1}$

57. $\lim_{x\to 0} (1+x)^{1/x} = ?$

Use ln as in 46.

$$\ln(1+x)^{1/x} = \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}.$$

for $\lim_{x\to 0} \frac{\ln(1+x)}{x}$, plugging in gives $\frac{0}{0}$. Use L'Hospital.

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\left(\frac{1}{1+x}\right)}{1} = 1.$$

So $\lim_{x\to 0} (1+x)^{1/x} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = e^1 = \boxed{e}$ ✓

Using a calculator, for example $x = 0.0007$ gives

$$\left|(1+x)^{1/x} - e\right| \leq 0.001$$

Section 4.6

1.

		f'	f''
A	decreasing, concave up	$-$	$+$
B	increasing, concave up	$+$	$+$
C	increasing, concave down	$+$	$-$
D	decreasing, concave down	$-$	$-$
E	decreasing, concave up	$-$	$+$
F	increasing, concave up	$+$	$+$
G	increasing, concave down	$+$	$-$

12. Sketch $y = x^{2/3}$ Domain: $\mathbb{R}$

$y' = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$ Never 0, but y' doesn't exist when $x = 0$.

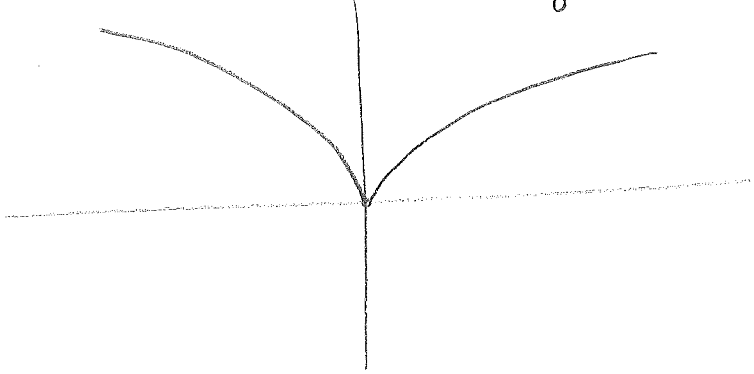
(Tan. line is vertical since $\lim_{x \to 0} y' = \infty$).

	$(-\infty, 0)$	0	$(0, \infty)$
y'	$-$ ↘		$+$ ↗

$y'' = \frac{-2}{9}x^{\frac{-4}{3}} = \frac{-2}{9x^{4/3}}$ This is never 0, but y'' doesn't exist when $x = 0$.

	$(-\infty, 0)$	0	$(0, \infty)$
y''	$-$ ⌒		$-$ ⌒

$y(0) = 0$



Decreasing and concave down on $(-\infty, 0)$

and increasing and concave down on $(0, \infty)$.

Derivative doesn't exist at $x = 0$.

12 cont. $y = x^{4/3}$ Domain: $\mathbb{R}$

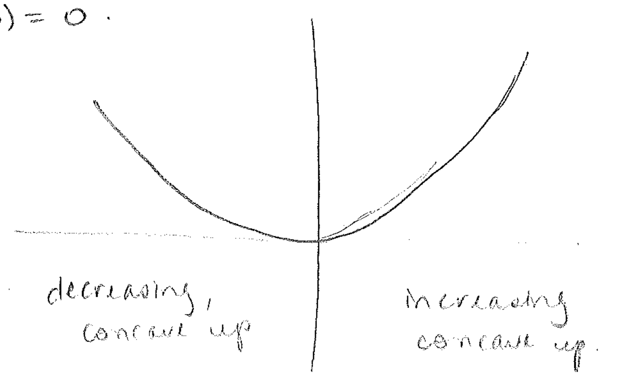
$y' = \frac{4}{3}x^{1/3} = 0$ when $x = 0$.

$y'' = \frac{4}{9}x^{-2/3} = \frac{4}{9x^{2/3}}$ which is never 0, but doesn't exist at $x = 0$

	$(-\infty, 0)$	0	$(0, \infty)$
y'	$-$		$+$

	$(-\infty, 0)$	0	$(0, \infty)$
y''	$+$		$+$

$y(0) = 0$.



14. $y = x^3 - 3x + 5$

$y' = 3x^2 - 3 = 3(x^2 - 1) = 0$ when $x = \pm 1$.

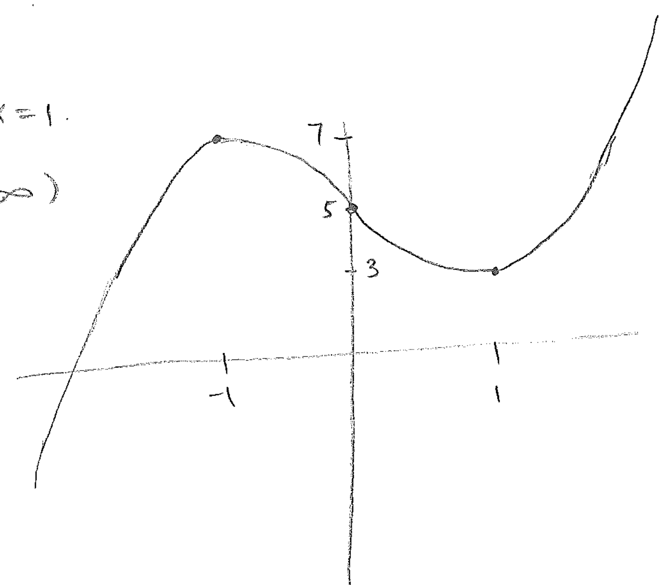
$y'' = 6x = 0$ when $x = 0$.

	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \infty)$
y'	$+ \nearrow$		$- \searrow$		$- \searrow$		$+ \nearrow$
y''	$- \frown$		$- \frown$		$+ \smile$		$+ \smile$

Transition points: $x = -1$, $x = 0$, $x = 1$.

y increasing on $(-\infty, -1)$, $(1, \infty)$
decreasing on $(-1, 1)$

y concave up on $(0, \infty)$
concave down on $(-\infty, 0)$



$y(0) = 5$

$y(-1) = -1 + 3 + 5 = 7$

$y(1) = 1 - 3 + 5 = 3$

$\lim_{x \to \infty} y = \infty$ and $\lim_{x \to -\infty} x^3 - 3x + 5 = -\infty$.

31. $y = x - 2\ln x$ Domain: $x > 0$ ($\ln x$ is only defined for $x > 0$).

$y' = 1 - \frac{2}{x}$

on the domain $x > 0$, $y' = 0$ only when $x = 2$ and y' always exists.

$y'' = \frac{2}{x^2}$

on the domain $x > 0$, y'' always exists and is never 0.

	$(0, 2)$	$(2, \infty)$
y'	$y'(1) = 1 - \frac{2}{1} = -1$ $-$ ↘	$y'(3) = 1 - \frac{2}{3} = \frac{1}{3}$ ↗ $+$
y''	$y''(1) = \frac{2}{1} = 1$ $+$ ∪	$y''(3) = \frac{2}{9}$ ∪ $+$

local min at $x = 2$:
$y(2) = 2 - 2\ln 2$
$\ln 2 < 1$ since $2 < e$,
so $y(2) > 0$

So y is increasing on $(2, \infty)$, and y is decreasing on $(0, 2)$.
y is concave up on $(0, \infty)$ and never concave down
There is a transition point at $x = 2$.

Check asymptotic behavior:

$\lim_{x \to \infty} x - 2\ln x = ?$

To compare the sizes of x and $\ln x$, consider $\frac{\ln x}{x}$

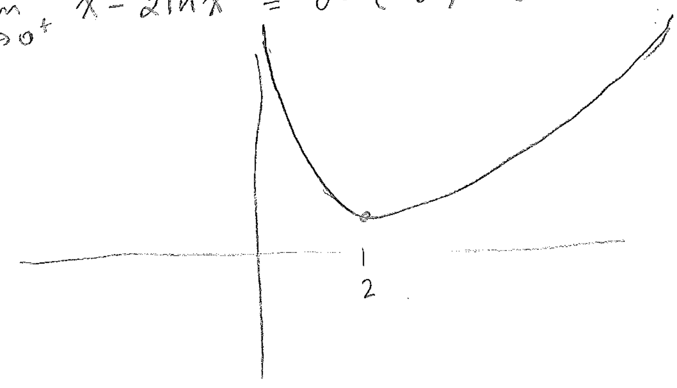
$$\lim_{x \to \infty} \frac{\ln x}{x} \underset{\uparrow}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

L'Hospital's rule

How does this help?

$$\lim_{x \to \infty} x - 2\ln x = \lim_{x \to \infty} x \Big(1 - \underbrace{\frac{2\ln x}{x}}_{\to 0}\Big) = \infty \cdot 1 = \infty$$

Also, $\lim_{x \to 0^+} x - 2\ln x = 0 - (-\infty) = \infty$. (Vertical asymptote at $x = 0$)



42. $y = \sin x + \cos x \quad [0, 2\pi]$

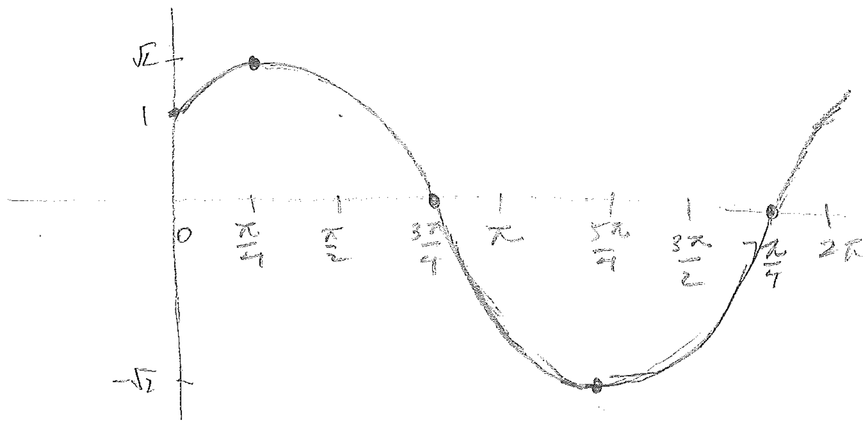
$y' = \cos x - \sin x \;=0$ when $\sin x = \cos x$, i.e. when $x = \frac{\pi}{4}, \frac{5\pi}{4}$

$y'' = -\sin x - \cos x \;=0$ when $\sin x = -\cos x$, i.e. when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

	0 to $\frac{\pi}{4}$	$\frac{\pi}{4}$ to $\frac{3\pi}{4}$	$\frac{3\pi}{4}$ to $\frac{5\pi}{4}$	$\frac{5\pi}{4}$ to $\frac{7\pi}{4}$	$\frac{7\pi}{4}$ to 2π
y'	$+ \nearrow$	$- \searrow$	$- \searrow$	$+ \nearrow$	$+ \nearrow$
y''	$- \cap$	$- \cap$	$+ \cup$	$+ \cup$	$- \cap$

$y(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

$y(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$



50. A is the graph of $f(x) = \frac{2x^4 - 1}{1 + x^4}$ since f has horizontal asymptotes at $y = 2$.

52. (a) (D) ($\lim_{x \to \pm\infty} \frac{1}{x^2-1} = 0$ and ± 1 are not in the domain. Also, $\frac{1}{x^2-1}$ is even)

(b) (A) ($\lim_{x \to \pm\infty} \frac{x^2}{x^2+1} = 1$)

(c) (B) (domain is $\mathbb{R}$ and $\lim_{x \to \pm\infty} \frac{1}{x^2+1} = 0$)

(d) (C) ($\lim_{x \to \pm\infty} \frac{x}{x^2-1} = 0$, ± 1 are not in the domain, and $\frac{x}{x^2-1}$ is an odd function).

58. $y = \frac{1}{x} - \frac{1}{x-1}$ Domain: $x \neq 0,\ x \neq 1$

$y' = -\frac{1}{x^2} - \frac{-1}{(x-1)^2} = 0$ when $\frac{1}{(x-1)^2} = \frac{1}{x^2}$, i.e. when

$0\ \cancel{x^2} = \cancel{x^2} - 2x + 1 \Rightarrow x = \frac{1}{2}$.

$y'' = \frac{2}{x^3} - \frac{2}{(x-1)^3} = 0$ when $\frac{2}{x^3} = \frac{2}{(x-1)^3}$, i.e. when

$0\ \cancel{x^3} = \cancel{x^3} - 3x^2 + 3x - 1$

$3x^2 - 3x + 1 = 0$ when $x = \underbrace{\frac{3 \pm \sqrt{9-12}}{6}}_{\text{no real roots.}}$

	$x<0$	$0<x<\frac{1}{2}$	$\frac{1}{2}<x<1$	$x>1$
y'	$y'(-1) = \frac{-1}{1} + \frac{1}{4} = -\frac{3}{4}$ $-$ ↘	$y'(\frac{1}{4}) = -16 + \frac{16}{9}$ $-$ ↘	$y'(\frac{3}{4}) = -\frac{16}{9} + 16$ $+$ ↗	$y'(2) = -\frac{1}{4} + \frac{1}{1}$ $+$ ↗
y''	$y''(-1) = -2 - \frac{2}{8}$ $-\ \cap$	$y''(\frac{1}{2}) = 16 + 16$ $+\ \cup$	$+\ \cup$	$y''(2) = \frac{1}{4} - 2$ $-\ \cap$

$\lim_{x \to \infty} \left(\frac{1}{x} - \frac{1}{x-1}\right) = 0 - 0 = 0$

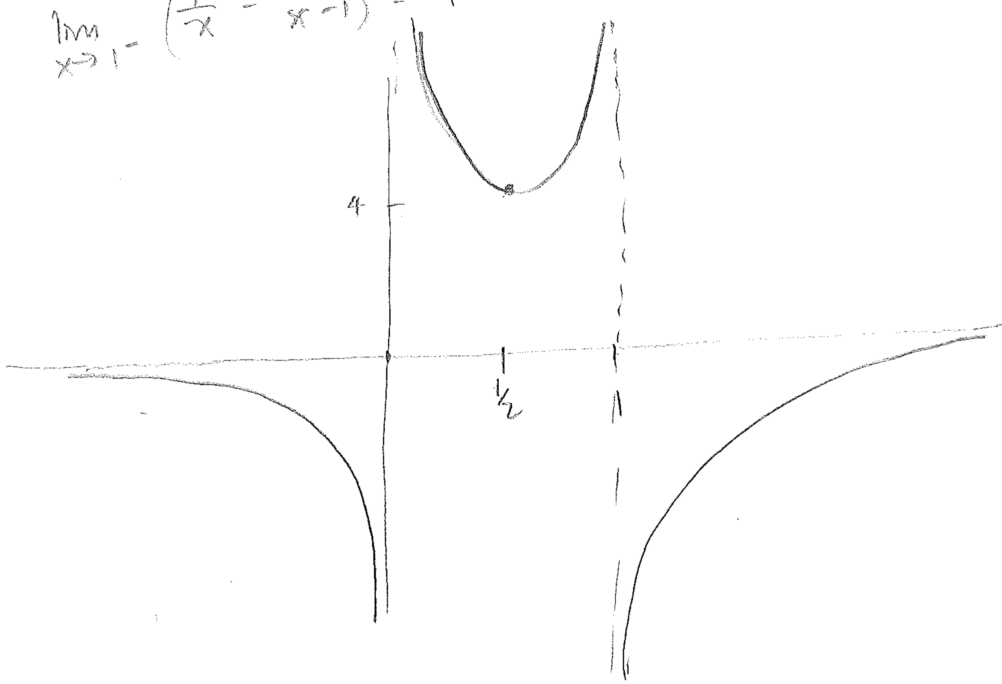
$\lim_{x \to -\infty} \left(\frac{1}{x} - \frac{1}{x-1}\right) = 0 - 0 = 0$

$\lim_{x \to 0^+} \frac{1}{x} - \frac{1}{x-1} = \infty - (-1) = \infty$

$\lim_{x \to 0^-} \frac{1}{x} - \frac{1}{x-1} = -\infty - (-1) = -\infty$

$\lim_{x \to 1^+} \left(\frac{1}{x} - \frac{1}{x-1}\right) = 1 - \infty = -\infty$

$\lim_{x \to 1^-} \left(\frac{1}{x} - \frac{1}{x-1}\right) = 1 - (-\infty) = \infty$



68. $y = \dfrac{x^2}{(x^2-1)(x^2+1)}$ Domain: $x \neq \pm 1$

$$y' = \frac{(x^2-1)(x^2+1)\cdot 2x - x^2\left(2x(x^2+1) + 2x(x^2-1)\right)}{(x^2+1)^2(x^2-1)^2}$$

$$= \frac{(x^4-1)2x - 2x^3(\cancel{2}x^2)}{(x^2+1)^2(x^2-1)^2} = \frac{2x^5 - 2x - 4x^5}{(x^2+1)^2(x^2-1)^2} = \frac{-2x(1+x^4)}{(x^2+1)^2(x^2-1)^2}$$

		-1		0		1	
y'	+		+		−		−

$$y'' = \frac{(x^2+1)^2(x^2-1)^2\left[-2-10x^4\right] + 2x(1+x^4)\left[2(x^2+1)\cdot 2x(x^2-1)^2 + 2(x^2-1)(x^2+1)^2 2x\right]}{(x^2+1)^4(x^2-1)^4}$$

$$= \frac{\cancel{(x^2+1)}\cancel{(x^2-1)}\left[(x^2+1)(x^2-1)(-2-10x^4) + 2x(1+x^4)\left[4x(x^2-1)+4x(x^2+1)\right]\right]}{(x^2+1)^{\cancel{4}3}(x^2-1)^{\cancel{4}3}}$$

$$= \frac{(x^4-1)(-2-10x^4) + 2x(1+x^4)\,8x^3}{(x^2+1)^3(x^2-1)^3} = \frac{6x^8+24x^4+2}{(x^2+1)^3(x^2-1)^3}$$

(Numerator of y'') > 0 for all x. Denominator > 0 when $x > 1$ or $x < -1$

Denominator < 0 when $-1 < x < 1$

		-1		1	
y''	+		−		+

$$\lim_{x\to\infty} \frac{x^2}{(x^2-1)(x^2+1)} = \lim_{x\to\infty} \frac{\frac{x^2}{x^4}}{\frac{1}{x^2}(x^2-1)(x^2+1)\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\left(1-\frac{1}{x^2}\right)\left(1+\frac{1}{x^2}\right)} = 0$$

Similarly, $\displaystyle\lim_{x\to-\infty} \frac{x^2}{(x^2-1)(x^2+1)} = 0.$

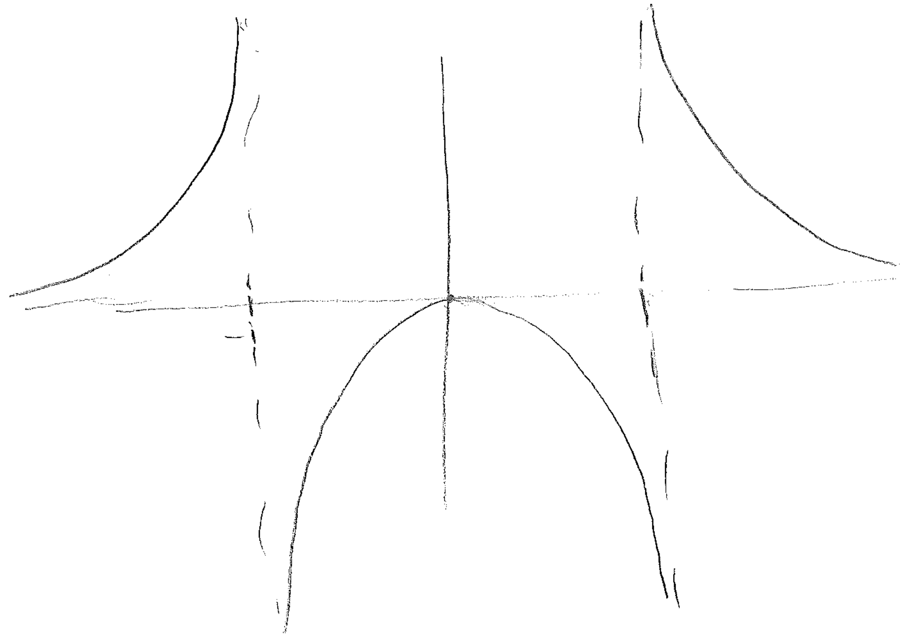
68 cont.

$$\lim_{x \to 1^+} \frac{x^2}{(x^2-1)(x^2+1)} = \infty$$ since if $x > 1$, $y > 0$. and the denominator is going to 0.

$$\lim_{x \to 1^-} \frac{x^2}{(x^2-1)(x^2+1)} = -\infty$$

$$\lim_{x \to -1^+} \frac{x^2}{(x^2-1)(x^2+1)} = -\infty$$

$$\lim_{x \to -1^-} \frac{x^2}{(x^2-1)(x^2+1)} = \infty$$



70. $y = \dfrac{x}{\sqrt{x^2+1}}$ Domain: $\mathbb{R}$.

$$y' = \frac{\sqrt{x^2+1} - x \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot \cancel{2}x}{x^2+1} = \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1}$$

$$= \frac{1}{(x^2+1)^{3/2}} \neq 0 \text{ for any } x$$

$$y'' = -\frac{3}{2}(x^2+1)^{-5/2} \cdot 2x = \frac{-3x}{(x^2+1)^{5/2}} = 0 \text{ when } x = 0$$

		0	
y'	+ ↗		+ ↗
y''	+ ∪		− ∩

70 continued:

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \to \infty} \frac{\frac{1}{x} \cdot x}{\frac{1}{x}\sqrt{x^2+1}} = \lim_{x \to \infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1$$

$$\lim_{x \to -\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \to -\infty} \frac{\frac{1}{x}\, x}{\frac{1}{x}\sqrt{x^2+1}} = \lim_{x \to -\infty} \frac{1}{-\sqrt{\frac{1}{x^2}}\sqrt{x^2+1}}$$

$$= \lim_{x \to -\infty} \frac{1}{-\sqrt{1+\frac{1}{x^2}}} = -1$$