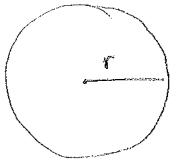
Homework 10 (Due 11/15)
Solutions

Section 3.11

3.



$r(t)$: radius of the oil slick at time t (in minutes)

Given: $\frac{dr}{dt} = 2$ m/min

Let $A(t)$ = area of the oil slick at time t.

a) $A(t) = \pi\, r(t)^2$

$\frac{dA}{dt} = 2\pi\, r(t) \cdot \frac{dr}{dt}$

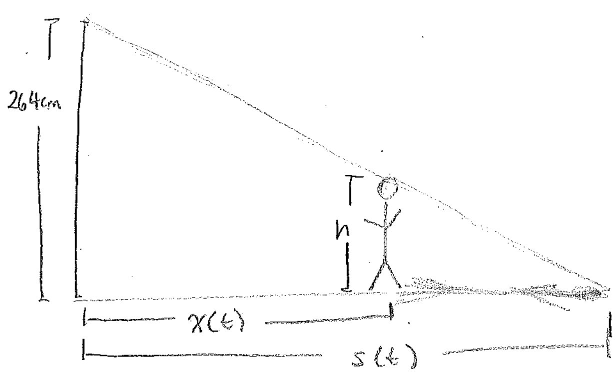
When $r = 25$m, $\frac{dA}{dt} = 2\pi \cdot 25 \cdot 2 = \boxed{100\pi \text{ m}^2/\text{min}}$

b) Given $r(0) = 0$

At $t = 3$ min, $r = 6$m since the radius is growing at 2m/min.

So $\left.\frac{dA}{dt}\right|_{t=3} = 2\pi \cdot 6 \cdot 2 = \boxed{24\pi \text{ m}^2/\text{min}}$

18.



$x(t)$ = distance between Claudia and the lamppost

$s(t)$ = distance between the tip of the shadow and the lamppost

$x'(t)$ = Claudia's speed.

$s'(t)$ = speed of the tip of her shadow.

h = Claudia's height (constant)

Given: $s'(t) = 2x'(t)$

Using similar triangles:

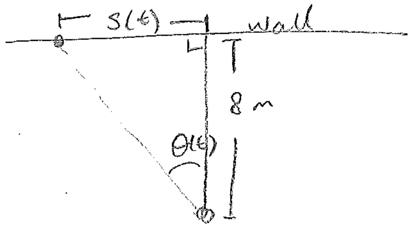
$$\frac{s(t) - x(t)}{h} = \frac{s(t)}{264}$$

Differentiate: $\frac{1}{h}\left(\frac{ds}{dt} - \frac{dx}{dt}\right) = \frac{1}{264}\frac{ds}{dt}$

$$\frac{1}{h}\left(2\frac{dx}{dt} - \frac{dx}{dt}\right) = \frac{1}{264} \cdot 2\frac{dx}{dt}$$

$$\frac{dx}{dt} \cdot \frac{264}{2} = h \cdot \frac{dx}{dt} \implies \boxed{h = 132 \text{ cm}}$$

22.



θ(t) = angle between beam from laser and the line through the searchlight at time t

s(t) = distance from searchlight beam to the beam from the laser at time t.

Given: Platform makes 20 rotations/min.

$\frac{ds}{dt}$ = speed of the dot on the wall.

Want $\frac{ds}{dt}$ when $\theta(t) = \frac{\pi}{6}$.

$$\frac{s(t)}{8} = \tan \theta(t)$$

Differentiate: $\frac{1}{8}\frac{ds}{dt} = \sec^2(\theta(t)) \cdot \frac{d\theta}{dt}$

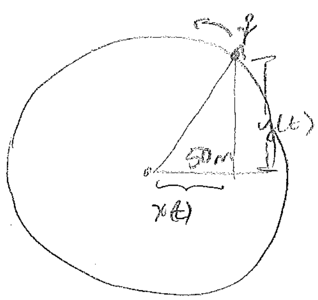
For $\frac{d\theta}{dt}$: 20 revolutions/min = $20 \cdot 2\pi$ radians/min.

$= 40\pi$

So $\frac{ds}{dt} = 8 \cdot \sec^2\left(\frac{\pi}{6}\right) \cdot 40\pi$

$= 320\pi \cdot \left(\frac{2}{\sqrt{3}}\right)^2 = \boxed{\frac{1280\pi}{3} \text{ m/min.}}$

29.



Let x(t) = Julian's x-coord at time t.
y(t) = Julian's y-coord at time t.

Given: When he is at (40, 30), $\frac{dx}{dt} = -1.25$ m/s.

Want: $\frac{dy}{dt}$ at this time.

$$x(t)^2 + y(t)^2 = 50^2$$

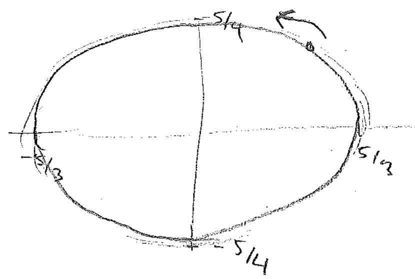
$$2x(t) \cdot \frac{dx}{dt} + 2y(t)\frac{dy}{dt} = 0$$

When $x(t) = 40$ and $y(t) = 30$ and $-1.25 = \frac{dx}{dt}$, plug in to get

$$80 \cdot (-1.25) + 60\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{100}{60} = \boxed{\frac{5}{3} \text{ m/sec}}$$

30. $9x^2 + 16y^2 = 25$



a) $\frac{dx}{dt}$ is positive when the x value is increasing, ie. in the third and fourth quadrants.

b) To get a relationship between $\frac{dx}{dt}$ and $\frac{dy}{dt}$, differentiate

$$9x^2 + 16y^2 = 25$$

$$\boxed{18x\frac{dx}{dt} + 32y\frac{dy}{dt} = 0}$$

c) When the particle passes $(1,1)$, $x=1$, $y=1$

If the y-coordinate is increasing at a rate of 6 m/sec

$\frac{dy}{dt} = 6$ at this time.

So $18 \cdot 1 \cdot \frac{dx}{dt} + 32 \cdot 1 \cdot 6 = 0$

and $\frac{dx}{dt} = \frac{-32 \cdot \cancel{6}}{\cancel{18}\,3} = \boxed{-\frac{32}{3} \text{ m/sec}}$

d) When the particle is at the top or bottom of the ellipse,

$x=0$, so

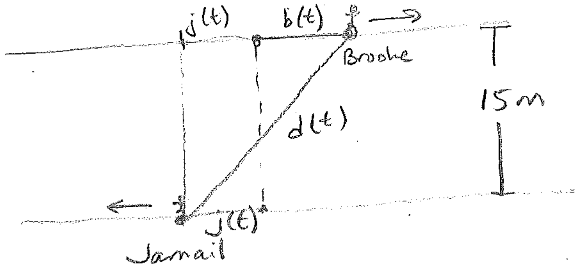
$$18 \cdot 0 \frac{dx}{dt} + 32y\frac{dy}{dt} = 0$$

$$32y\frac{dy}{dt} = 0$$

So $\boxed{\frac{dy}{dt} = 0}$ (since $y \neq 0$)

at the top and bottom of the ellipse.

(Alternately: At the top and bottom of the ellipse, y changes from increasing to decreasing or vice versa. That is, $\frac{dy}{dt}$ changes from positive to negative or vice versa. So $\frac{dy}{dt}$ must be 0 at the top and bottom of the ellipse.)

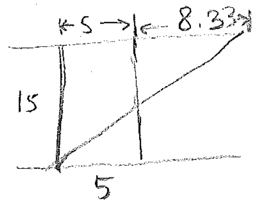
34. 

Set $b(t)$ = distance Brooke has gone since time $t=0$.
$j(t)$ = distance Jamail has gone since time $t=0$
$d(t)$ = distance between Brooke and Jamail.

Know: $b'(t) = 10\,\text{km/hr} = 10\,\frac{\text{km}}{\text{hr}} \times 1000\,\frac{\text{m}}{\text{km}} \times \frac{1\,\text{hr}}{60\,\text{min}} \times \frac{1\,\text{min}}{60\,\text{sec}} \approx 2.78\,\text{m/s}.$

$j'(t) = 6\,\text{km/hr} = 6\,\frac{\text{km}}{\text{hr}} \times 1000\,\frac{\text{m}}{\text{km}} \times \frac{1\,\text{hr}}{60\,\text{min}} \times \frac{1\,\text{min}}{60\,\text{sec}} \approx 1.67\,\text{m/s}$

After 3 seconds, $j(t) = 5$ m, and $b(t) = 8.33$ m.



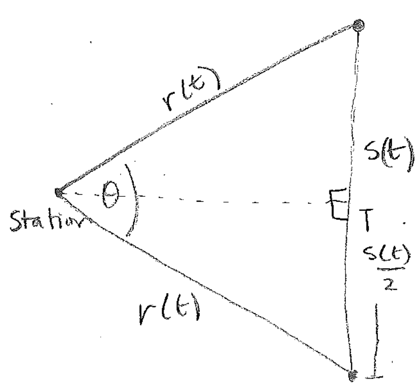
$$d(3) = \sqrt{15^2 + (5+8.33)^2} \approx \boxed{20.07\text{m}}$$

$$d(t) = \sqrt{15^2 + (j(t)+b(t))^2}$$

$$d'(t) = \frac{1}{2}\left(15^2 + (j(t)+b(t))^2\right)^{-1/2} \cdot \left(2\,(j(t)+b(t))\right)\cdot\left(j'(t)+b'(t)\right).$$

So at $t=3$:

$$d'(3) = \cancel{\frac{1}{2}} \cdot \frac{1}{20.07} \cdot \cancel{2}\,(5+8.33)\,(1.67+2.78) \approx \boxed{2.95\,\text{m/s}}$$

42. 

$r(t)$ = distance traveled by the trains.
Given: $r'(t) = v$ (constant)
$s(t)$ = distance between the trains.
Want: $s'(t)$.

$$\frac{s(t)/2}{r(t)} = \sin\frac{\theta}{2} \implies \frac{s(t)}{2} = r(t)\sin\frac{\theta}{2}$$

a) So $\frac{1}{2}s'(t) = \left(\sin\frac{\theta}{2}\right)\cdot r'(t) \to s'(t) = 2v\sin\frac{\theta}{2}$ ✓
$= 2v\sqrt{\frac{1-\cos\theta}{2}} = v\sqrt{2-2\cos\theta}$

34 b) when $\theta = \pi$,

$$\frac{ds}{dt} = v \cdot \sqrt{2 - 2\cos\pi} = v \cdot \sqrt{2}.$$

Section 4.1

4. $f(x) = \dfrac{1}{x+1}$

Estimate $f(3.02) - f(3)$:

$$f(3.02) - f(3) \approx f'(3) \cdot \Delta x = f'(3)(0.02)$$

$$f'(x) = \frac{-1}{(x+1)^2}, \text{ so } f'(3) = \frac{-1}{(4)^2} = \frac{-1}{16}.$$

$$f(3.02) - f(3) \approx \frac{-1}{16} \cdot (0.02) = \boxed{-0.00125}$$

5. $f(x) = \sqrt{x+6}$

Estimate $f(3.02) - f(3)$:

$$f(3.02) - f(3) \approx f'(3) \cdot \Delta x = f'(3) \cdot (0.002).$$

$$f'(x) = \frac{1}{2}(x+6)^{-1/2}, \text{ so } f'(3) = \frac{1}{2} \cdot 9^{-1/2} = \frac{1}{6}.$$

$$f(3.02) - f(3) \approx \frac{1}{6}(0.002) = \boxed{\frac{1}{3000}}$$

30. $F(s) = 1.1s + 0.054s^2$ ft (s = speed in mph)

$$\Delta F \approx F'(s) \cdot \Delta s.$$

$$F'(s) = 1.1 + 0.108s$$

When $s = 35$: $\Delta F \approx [1.1 + 0.108(35)] \cdot (1) = \boxed{4.88 \text{ ft}}$ ($\Delta s = 1$ since we are thinking about the effect of adding one mph).

When $s = 55$: $\Delta F \approx [1.1 + 0.108(55)](1) = \boxed{7.04 \text{ ft}}$

(This says that if you're going 36 mph instead of 35 mph, your stopping distance increases by about 4.88 ft and if you're going 56 mph instead of 55 mph, your stopping distance increases by about 7.04 ft)

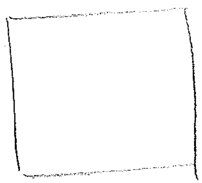
31. $L = 18$ cm when $T = 30°C$.

Estimate ΔL when T decreases to $25°C$ if $k = 1.2 \times 10^{-5}$ $°C^{-1}$.

Know (from example 3 of the book): $\frac{dL}{dT} = kL$.

$$\Delta L \approx L'(T)\,\Delta T = (1.2\times 10^{-5})(18)(-5°C)$$
$$= \boxed{-0.00108 \text{ cm}}$$

38.



$s = 6$m

$s^2 = A$

s accurate to within 2 cm means Δs could be up to 2cm.

$$\Delta A \approx A'(s)\cdot \Delta s$$
$$= 2s\,\Delta s$$

$$\Delta A \approx (2)(6)(0.02) = \boxed{0.24 \text{ m}^2}$$

46. $f(x) = \frac{1}{x}$ $\quad a = 2$

Slope: $f'(x) = \frac{-1}{x^2}$, so $f'(a) = f'(2) = \frac{-1}{4}$.

Point: $f(2) = \frac{1}{2}$, so our point of tangency is $(2, \frac{1}{2})$.

$$\boxed{L(x) = -\frac{1}{4}(x-2) + \frac{1}{2}}$$

52. $y = \tan^{-1} x$, $\quad a = 1$.

Slope: $y' = \frac{1}{x^2+1}$, so $y'(1) = \frac{1}{2}$.

Point: $y(1) = \tan^{-1}(1) = \frac{\pi}{4}$, so the point of tangency is $(1, \frac{\pi}{4})$.

$$\boxed{L(x) = \frac{1}{2}(x-1) + \frac{\pi}{4}}$$

71. $y^3 + 3xy = 7$, $\quad P = (2,1)$.

Use implicit differentiation to find dy/dx.

$$3y^2 \frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 0$$

At $(2,1)$: $\quad 3\cdot(1)^2 \frac{dy}{dx} + 3(2)\frac{dy}{dx} + 3(1) = 0$

$$9\frac{dy}{dx} = -3, \text{ so } \boxed{\frac{dy}{dx} = \frac{-1}{3}}$$

71 cont. So the tangent line to the curve at $(2,1)$ has equation

$$y-1 = \frac{-1}{3}(x-2) \quad \text{or} \quad y = \frac{-1}{3}(x-2)+1.$$

Thus, $L(x) = \frac{-1}{3}x + \frac{2}{3} + 1 = \boxed{\frac{-1}{3}x + \frac{5}{3}}$ ✓

When $x = 2.1$, $y \approx L(2.1) = \frac{-1}{3}(2.1) + \frac{5}{3} = \boxed{0.967}$

Section 4.2.

1. a) $f(x)$ has 3 critical points on $[0,8]$. (Namely $x=3$, $x=5$, $x=7$).

b) The maximum value of $f(x)$ on $[0,8]$ is 6.

c) $f(x)$ has a local maximum value of 5 at $x=5$.

d) $f(x)$ takes both min and max values at critical points on the interval $[2,6]$. (or $[4,8]$, or others).

e) The minimum value of $f(x)$ on the interval $[0,2]$ occurs at an endpoint (at 2).

3. $f(x) = x^2 - 2x + 4$

$f'(x) = 2x - 2$.

Critical points: $2x - 2 = 0$

$\boxed{x=1}$

8. $g(z) = \frac{1}{z-1} - \frac{1}{z}$.

$$g'(z) = \frac{-1}{(z-1)^2} - \frac{-1}{z^2} = \frac{-z^2 + (z-1)^2}{z^2(z-1)^2} = \frac{-2z+1}{z^2(z-1)^2}$$

Critical point: $\frac{-2z+1}{z^2(z-1)^2} = 0$:

$-2z+1 = 0$

So $\boxed{z = \frac{1}{2}}$

12. $f(t) = 4t - \sqrt{t^2+1}$.

$$f'(t) = 4 - \frac{1}{2}(t^2+1)^{-1/2} \cdot 2t$$

$$= 4 - \frac{t}{\sqrt{t^2+1}}$$

Critical points: $4 - \frac{t}{\sqrt{t^2+1}} = 0$

$$(4)^2 = \left(\frac{t}{\sqrt{t^2+1}}\right)^2$$

$$16 = \frac{t^2}{t^2+1}, \text{ so } 16t^2+16 = t^2$$

$$15t^2 = -16$$

$$t^2 = \frac{-16}{15}$$ No real solutions t, so f has $\boxed{\text{no critical points}}$

16. $R(\theta) = \cos\theta + \sin^2\theta$.

$$R'(\theta) = -\sin\theta + 2\sin\theta\cos\theta.$$

Critical points:

$$0 = -\sin\theta + 2\sin\theta\cos\theta = \sin\theta(2\cos\theta - 1).$$

Either $\sin\theta = 0 \Rightarrow \theta = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$

or $2\cos\theta - 1 = 0 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots$

So the critical points are

$$\boxed{\theta = n\pi, \quad \theta = \frac{\pi}{3} + 2\pi n, \quad \theta = \frac{5\pi}{3} + 2\pi n \text{ for any integer } n.}$$

17. $f(x) = x\ln x$

$$f'(x) = x \cdot \frac{1}{x} + \ln x. = 1 + \ln x.$$

Critical points: $\ln x + 1 = 0$

$$\ln x = -1$$

$$\boxed{x = e^{-1} = \frac{1}{e}.}$$

22. $f(x) = 2x^3 - 9x^2 + 12x$

Find extreme values on $[0,3]$ and $[0,2]$.

$f'(x) = 6x^2 - 18x + 12$.

Critical points: $6x^2 - 18x + 12 = 0$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$, so $x = 2$, $x = 1$.

On $[0,3]$: Check critical points in this range:

$f(2) = 2 \cdot 8 - 9 \cdot 4 + 24 = 4$.

$f(1) = 2 - 9 + 12 = 5$.

Check end points:

$f(0) = 0$.

$f(3) = 2 \cdot 27 - 9 \cdot 9 + 36 = 9$.

Max value: $f(3) = 9$.
Min value: $f(0) = 0$.

On $[0,2]$: Check critical points in this range

$f(1) = 5$

Check endpoints:

$f(0) = 0$

$f(2) = 4$.

Max. value: $f(1) = 5$
Min value: $f(0) = 0$

23. $f(x) = \sin x + \cos x$. Determine extreme values on $[0, \frac{\pi}{2}]$.

$f'(x) = \cos x - \sin x$.

Critical points: $\cos x - \sin x = 0 \Rightarrow \sin x = \cos x$

on $[0, \frac{\pi}{2}]$, the only solution to $\sin x = \cos x$ is $x = \frac{\pi}{4}$.

Check critical points:

$f(\frac{\pi}{4}) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$.

Check end points:

$f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$

$f(\frac{\pi}{2}) = \sin \frac{\pi}{2} + \cos \frac{\pi}{2} = 1 + 0 = 1$.

Max value: $\sqrt{2}$
Min. value: 1