Class 9: 10/4

Outline
I. Orthogonality of 4 subspaces
II. Orthogonal complement
III. Projections
Collect HW. Last time: F.T.L.A part 1

Go back to orthogonality (notes from last time).

So $N(A) \perp C(A^T)$

Note: This implies $N(A^T) \perp C(A)$ so left nullspace and column space are orthogonal.

Some facts about orthogonal complement:

If V is a subspace of $\mathbb{R}^n$, then
1) $\dim V^\perp + \dim V = n$

2) Any vector $\vec{v}$ in $\mathbb{R}^n$ can be written uniquely as $\vec{v}_1 + \vec{v}_2$ where $\vec{v}_1 \in V$, $\vec{v}_2 \in V^\perp$

3) If $\vec{v} \in V$ and in $V^\perp$, $\vec{v} = \vec{0}$.

4) $(V^\perp)^\perp = V$
(Use if $W \subseteq V$, and $\dim W = \dim V$, then $W = V$).

So any $\vec{x}$ in $\mathbb{R}^n = \underset{\text{in row space}}{\vec{x}_r} + \underset{\text{in null space}}{\vec{x}_n}$

$$A\vec{x} = A(\vec{x}_r + \vec{x}_n) = A\vec{x}_r + A\vec{x}_n = A\vec{x}_r.$$

* Everything in the column space comes from a vector in the row space.

Also, everything in the column space comes from a <u>unique</u> vector in the row space.

Why? If $A\vec{x}_r = A\vec{x}_r'$, then $A(\vec{x}_r - \vec{x}_r') = 0$.

So $\vec{x}_r - \vec{x}_r'$ is in the nullspace and in the row space

Since $N(A) \perp C(A^T)$, $x_r - x_r' = 0$, and $x_r = x_r'$.

<u>Conclusion</u>: A as a function from row space to column space is <u>invertible</u>.

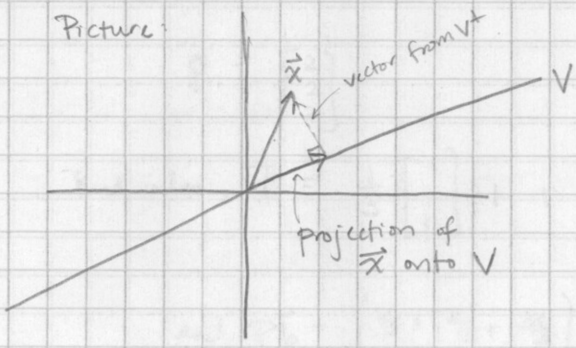
A has an $r \times r$ invertible submatrix.

Preview of projections:

- Any vector in $\mathbb{R}^n$ can be written uniquely as a sum of a vector in V and a vector in $V^\perp$

The projection of $\vec{x}$ onto V is the vector $\vec{v} \in V$ with
$$\vec{x} = \vec{v} + \vec{w}, \quad \vec{v} \in V, \quad \vec{w} \in V^\perp.$$

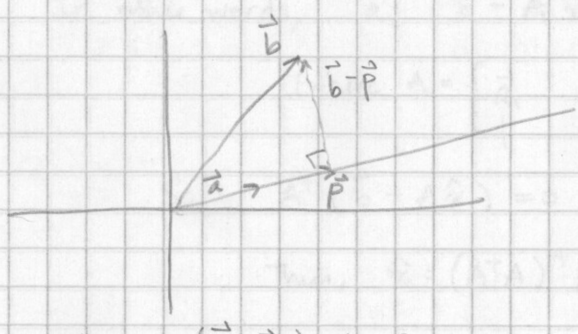
Picture:



Projection onto a line

Say V is a line, with basis $\vec{a}$, and we want to project $\vec{b}$ onto V.

If the projection is $\vec{p}$, $\vec{p} = c\vec{a}$ for some c. (constant). Find c.



Key: $(\vec{b}-\vec{p}) \perp \vec{a}$

So $\vec{a} \cdot (\vec{b} - c\vec{a}) = 0$

$$\vec{a}\cdot\vec{b} - c\,\vec{a}\cdot\vec{a} = 0$$

$$c = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}^T\vec{a}} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}$$
$$= \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}$$

Note: $\vec{p} = \vec{a} \cdot \left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right) = \underbrace{\frac{\vec{a}\cdot\vec{a}^T}{\vec{a}^T\vec{a}}}_{\text{matrix } P} \vec{b}$

Example: Project $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the line spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\vec{p} = \hat{x}\,\vec{a} \quad \text{where } \hat{x} = \frac{(1,\ 1,\ 1)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{3} = \frac{6}{3} = 2$$

$$\vec{p} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

But also, $P = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

and $P\vec{b} = \begin{bmatrix} 1/3 + 2/3 + 3/3 \\ 1/3 + 2/3 + 3/3 \\ 1/3 + 2/3 + 3/3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

<u>Next</u>: Project onto a space V with basis $\vec{a}_1, \vec{a}_2, \cdots \vec{a}_n$.

Need $\vec{b} - (\hat{x}_1 \vec{a}_1 + \cdots + \hat{x}_n \vec{a}_n) \perp \vec{a}_i$ for every i.

In other words, $\vec{a}_i^{\,T}(\vec{b} - A\hat{x}) = 0$ for $i = 1, 2, \ldots, n$.

(where $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$)

So $A^T(\vec{b} - A\hat{x}) = 0 \implies \boxed{A^T\vec{b} = A^TA\hat{x}}$

Thus, $\hat{x} = (A^TA)^{-1}A^T\vec{b}$ and $\boxed{\vec{p} = A(A^TA)^{-1}A^T\vec{b}}$

* Projection matrix

$$P = A(A^TA)^{-1}A^T$$

(Note: A^TA is invertible if and only if A has indep. columns)

<u>Definition</u>: A projection matrix is a matrix P s.t.

1) $P^T = P$

2) $P^2 = P$.