Class 8: 10/2

Announce: Grading of HW

Outline

I. Four subspaces: Dimension
Description
Orthogonality.

Last time: • All solutions to $A\vec{x} = \vec{b}$: particular solution $\vec{x}_p$ + nullspace
• Defined linear independence, basis, dimension.

Associated to a matrix A, we have 4 subspaces

1. Row space = span of row vectors = $C(A^T)$
2. Column space = Span of column vectors = $C(A)$
3. Nullspace = all vectors $\vec{x}$ so that $A\vec{x} = 0$ = $N(A)$
4. Left nullspace = all vectors $\vec{x}$ so that $\vec{x}A = 0$ = $N(A^T)$.

Today we'll describe a basis for each subspace and find dimension.

First, go back + cover full column rank, full row rank and solutions to $A\vec{x} = \vec{b}$ possibilities.

To describe the four subspaces, start with a matrix in reduced row echelon form.

E.g. $\left(A = \begin{bmatrix} 1 & 1 & 3 & -4 \\ 2 & 2 & 5 & -6 \\ 3 & 3 & 10 & -14 \end{bmatrix} \longrightarrow \right) R = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1. Row space of R is spanned by 1[ST] and 2[ND] rows.
   - the rows are independent (look at 1[ST] and 3[RD] columns).
   - third row adds nothing.

Fact: Row space of R is spanned by pivot rows, so dim = r

2. Column space of R is spanned by 1[ST] and 3[RD] columns
   - they are independent (e_1, e_2)
   - other columns are linear combos of these

Fact: Column space of R is spanned by pivot columns, so dim = r.

3. Nullspace is spanned by special solutions and in fact, special solutions form a basis.

Note: Why are special solutions independent?

Fact $\dim N(R) = \#$ free variables $= n - r$

4. Left nullspace: look for comb of rows to be 0.
But 1^{ST} & 2^{ND} rows are independent.

$$\begin{array}{rl} & a\,(1,\ 1,\ 0,\ 2) \\ + & b\,(0,\ 0,\ 1,\ -2) \\ + & c\,(0,\ 0,\ 0,\ 0) \\ \hline = & (0,\ 0,\ 0,\ 0) \end{array}$$

So $a = b = ?$ (0)
c is free.

Fact: $\dim N(R^T) = \#$ zero rows $= m - r$

Basis $= (\underbrace{0, 0, 0, \ldots, 0}_{r \text{ zeros}}, \underbrace{1, 0, 0 \ldots, 0}_{m-r})$, $(\underbrace{0, \ldots 0}_{r}, \underbrace{0, 1, 0 \ldots 0}_{m-r})$, ...

Now, what does this tell us about A?

1. Row space of A = Row space of R (HW problem)

Why? $EA = R$ means rows of R are combos of rows of A.
$A = E^{-1}R$ means rows of A are combos of rows of R.

Basis of $C(A^T)$ = pivot rows of R or pivot rows of A
$\dim C(A^T) = r$

2. Column space of A $\neq$ Column space of R.

But! $\dim C(A) = \dim C(R) = r$

Also elimination preserves column relations.
That is if $A\vec{x} = 0$, $R\vec{x} = 0$ and vice versa.
So if a column of A is a linear combo of other columns, the same relation holds in R.

So pivot columns are independent in both.

* Pivot columns of $\underline{\underline{A}}$ = basis for $C(A)$

3. $N(A) = N(R)$ (HW problem).

Same basis, same dimension

special solutions $\quad$ $n-r$.

4. Left nullspace: Need vectors $(x_1, x_2, \ldots, x_m)$ s.t.

$$(x_1, \ldots, x_m) A = \vec{0}$$

This is the same as asking for linear combos of rows of A that are $\vec{0}$.

When R has rows of zeros, this tells us that a linear combo of rows of $A = 0$!

$\#$ zero rows of $R = \dim N(A^T) = m-r$

Basis = last $m-r$ rows of E in $A = E^{-1}R$.

(Alternately, use that left nullspace is $N(A^T)$.
we know $\dim N(A) = n-r$, so $\dim N(A^T) = m-r$)

__Theorem__ (Fund. Thm of Linear Alg. Part 1)

$$\dim C(A) = \dim C(A^T) = \text{rank } r.$$

$$\dim N(A) = n-r$$

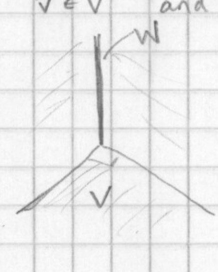
$$\dim N(A^T) = m-r$$

Part 2 of the theorem says that $\underset{\subset \mathbb{R}^m}{C(A)}$ is orthogonal (perpendicular) to $\underset{\subset \mathbb{R}^m}{N(A^T)}$ and $\underset{\subset \mathbb{R}^n}{C(A^T)}$ is orthogonal to $N(A) \subset \mathbb{R}^n$

__Definition__: Two vectors $\vec{v}, \vec{w}$ are __orthogonal__ if $\vec{v}^T \vec{w} = \vec{v} \cdot \vec{w} = 0$.

Two subspaces V and W are orthogonal if $\vec{v}^T \vec{w} = 0$ for all $\vec{v} \in V$ and $\vec{w} \in W$.

ex.



Fact: Nullspace and Row space are orthogonal.

Say $A\vec{x}=\vec{0}$. Then $\begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_m & - \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

So $\vec{v}_1 \cdot \vec{x} = 0$
$\vec{v}_2 \cdot \vec{x} = 0$
$\vdots$
$\vec{v}_m \cdot \vec{x} = 0$.

Upshot: $\vec{x}$ is orthogonal to every row of A.

But if $\vec{v} \in$ row space of A, $\vec{v} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$ (linear combo of rows)

Then $\vec{v}\cdot\vec{x} = c_1\vec{v}\cdot\vec{x} + c_2\vec{v}\cdot x + \dots + c_m\vec{v}\cdot x$
$= c_1\cdot 0 + c_2\cdot 0 + \dots + c_m\cdot 0 = 0$.

So $N(A) \perp C(A^T)$.

Same proof shows $N(A^T) \perp C(A)$.

In fact, something stronger is true. $N(A)$ and $C(A^T)$ are orthogonal complements. (So are $N(A^T)$ and $C(A)$).

Definition: The orthogonal complement of a subspace V is the set of all vectors perpendicular to V.

We write $V^\perp$ (say "V perp")

Fundamental Thm of Linear Alg. Part 2

$N(A)$ is the orthogonal complement of $C(A^T)$ (in $\mathbb{R}^n$)

$N(A^T)$ is the orthogonal complement of $C(A)$ (in $\mathbb{R}^m$)