Class 7: 9/27/12

Outline
I. Dimension, basis, independence
II. All solutions to $A\vec{x}=\vec{b}$

Last time: Nullspace = linear combinations of special solutions.
Rank = # pivots = dimension of column space.

Definition: A set of vectors $v_1, v_2, \ldots, v_n$ is linearly independent if

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0 \quad \text{only when } c_1 = c_2 = c_3 = \cdots = c_n = 0.$$

Alternate definition: Nullspace of $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$ is $Z = \{\vec{0}\}$.

When this happens, we say the columns of A are independent.

Geometrically: Vectors are independent if they "go in different directions." A set of n vectors is independent if all linear combinations span $\mathbb{R}^n$ (biggest space possible).

Example: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are independent

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are dependent

Note: If $n > m$, a set of n vectors in $\mathbb{R}^m$ is necessarily dependent

Definition: A set of vectors spans a space if all linear combinations of the vectors fill the space.

E.g. Columns of A span $C(A)$
Special solutions span $N(A)$.

Definition: Row space of A = span of row vectors (as seen on HW)

* Alternatively, row space = $C(A^T)$.

Note: Vectors don't have to be linearly independent to span a space

e.g. $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ span $\mathbb{R}^2$ even though $\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\end{bmatrix}$ alone already span $\mathbb{R}^2$.

If we don't have redundant vectors, we have a basis.

Definition: A basis of a vector space is a set of vectors that
1) span the space
2) are linearly independent.

E.g. $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are a basis for $\mathbb{R}^2$.

$\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ do not form a basis for $\mathbb{R}^2$.

Fact: If $v_1, v_2, \ldots, v_n$ is a basis for V, then there is only one way to write a vector in V as a combination of basis vectors.

Why? Say $a_1v_1 + a_2v_2 + \cdots + a_nv_n = b_1v_1 + \cdots + b_nv_n$.

Then $(a_1-b_1)v_1 + (a_2-b_2)v_2 + \cdots + (a_n-b_n)v_n = 0$.

Since the v_i are independent, $a_1 = b_1$
$a_2 = b_2$
$\vdots$
$a_n = b_n$

So the two linear combinations were actually the same.

Facts:
- Pivot columns of A form a basis for the column space (last time: free columns = combination of pivot columns, so those are redundant).
- Pivot rows of A form a basis for the row space.
- $v_1, \ldots, v_n$ form a basis for $\mathbb{R}^n$ if and only if the matrix $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & v_3 \cdots & v_n \\ | & | & & | \end{bmatrix}$ is an $n \times n$ invertible matrix.

Definition: Dimension of a vector space = # vectors in a basis.

II. Use our nullspace experience to find **all** solutions $A\vec{x}=\vec{b}$.

Note: If $\vec{x}_n \in N(A)$ and $\vec{x}_p$ is a vector with $A\vec{x}_p=\vec{b}$, then

$$A(\vec{x}_p+\vec{x}_n) = A\vec{x}_p + A\vec{x}_n = \vec{b}+\vec{0} = \vec{b}.$$

So we can add any vector in the nullspace to any particular solution to get another solution.

Definition: A matrix has **full column rank** if $r=n$

Then
1. All columns are pivot columns
2. No free variables or special solutions, so $N(A)=0$
3. If $A\vec{x}=\vec{b}$ has a solution, the solution is unique.
4. Columns are independent

Definition: A matrix has **full row rank** if $r=m$

Then
1. All rows have pivots, and R has no zero rows.
2. $A\vec{x}=\vec{b}$ is always solvable.
3. $C(A)=\mathbb{R}^m$
4. $N(A)$ has $n-r=n-m$ special solutions.
5. rows of A are independent

Possibilities:

1) $R=[I]$; A is square, invertible $A\vec{x}=\vec{b}$ has exactly 1 solution.
$r=m=n$

2) $R=[I \mid F]$: $A\vec{x}=\vec{b}$ has ∞-ly many solutions.
$r=m<n$

3) $R=\begin{bmatrix} I \\ \hline 0 \end{bmatrix}$ $A\vec{x}=\vec{b}$ has 0 or 1 solution.
$r=n<m$

4) $R=\left[\begin{array}{c|c} I & F \\ \hline 0 & 0 \end{array}\right]$ $A\vec{x}=\vec{b}$ has 0 or ∞-ly many solutions.
$r<m$, $r<n$

Example: Find *all* solutions to $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Eliminate $\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 2 \\ 2 & 2 & 8 & 10 & 1 \\ 3 & 3 & 10 & 13 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 2 \\ 0 & 0 & 4 & 4 & -3 \\ 0 & 0 & 4 & 4 & -3 \end{array}\right]$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 2 \\ 0 & 0 & 4 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc} 1 & 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 & -3/4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 7/2 \\ 0 & 0 & 1 & 1 & -3/4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Particular solution?

Choose values for free variables, say $x_2 = x_4 = 0$.

Then $x_1 = 7/2$ and $x_3 = -3/4$.

$\begin{bmatrix} 7/2 \\ 0 \\ -3/4 \\ 0 \end{bmatrix}$ is a solution.

All solutions are $\begin{bmatrix} 7/2 \\ 0 \\ -3/4 \\ 0 \end{bmatrix}$ + nullspace vectors

$N = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$ from $N = \begin{bmatrix} -F \\ I \end{bmatrix}$

So All solutions are

$$\begin{bmatrix} 7/2 \\ 0 \\ -3/4 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Note: If $A\vec{x}_1 = \vec{b}$ and $A\vec{x}_2 = \vec{b}$, then $\vec{x}_1 = \vec{x}_2 + \underbrace{\vec{x}_n}_{\text{in nullspace}}$ since

$A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = 0 \Rightarrow A(\vec{x}_1 - \vec{x}_2) = 0$, so $\vec{x}_1 - \vec{x}_2 \in N(A)$