Class 4: 9/18/12

<u>Outline</u>:

I. Elimination with Augmented matrices II Finding inverses (Gauss Jordan elim)

III. LU Decomposition

IV. Transpose + Permutations

* Hand back HW 1.

Last time: Elimination matrices E_{ij} = Identity matrix with the 0 in the ij place replaced by $-l$.

Acts by subtracting l times row j from row i

Permutation matrices P_{ij} = Identity matrix with i and j rows switched

Acts by switching rows i and j.

I. Augmented matrices.

↳ Some people used this on HW 1. Volunteers to explain?

<u>Problem</u>: Solve

$$\begin{aligned} x+2y+z &= 1 \\ 2x+3y+z &= 0 \\ -x+y+3z &= 0. \end{aligned}$$

1) Matrix?

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad (A\vec{x} = \vec{b})$$

2) Augment

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right] = [A|\vec{b}]$$

3) Eliminate. Let's keep track of elimination matrices

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ Acts to give } E_{21}[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & -2 \\ -1 & 1 & 3 & 0 \end{array}\right]$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +1 & 0 & 1 \end{bmatrix} \text{ Acts to give } E_{31}E_{21}[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 3 & 4 & 1 \end{array}\right]$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \text{ Acts to give } E_{32}E_{31}E_{21}[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

So $\quad \begin{aligned} x+2y+z &= 1 \\ -y-z &= -2 \\ z &= -5 \end{aligned}$ $\quad x=-8$, $\quad -y=-7 \rightarrow y=7$

Alternatively, use <u>Gauss</u>-<u>Jordan</u> Eli<u>minati</u>on to get this answer
(No longer need to back solve).

<u>Idea</u>: $\left[A \mid \vec{b}\right]$ gets multiplied by A^{-1} to give $\left[A \mid A^{-1}\vec{b}\right]$.
(Continue eliminating so we get only pivots).

<u>Recall</u>: Defn of inverse
Go back to facts about inverses.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right] \xrightarrow{E_{12}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right] \xrightarrow{E_{13}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -8 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$$\xrightarrow{E_{23}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -8 \\ 0 & -1 & 0 & -7 \\ 0 & 0 & 1 & -5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

We can use Gauss-Jordan elimination to find A^{-1}:

$$\left[A \mid I\right] \longrightarrow \left[I \mid A^{-1}\right]$$

In this example, $\begin{bmatrix} -8 \\ 7 \\ 5 \end{bmatrix}$ will be the first column of A^{-1}.

We are solving $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ all at once.

III. LU Decomposition:

Elimination factors matrices into LU
lower ↗ ↖ upper

In our example:

$$E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Product of the E's: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix}$.

Fact: For the inverse of E_{ij}, negate the ℓ (not the 1's on the diagonal)
Why? To reverse the process of subtracting ℓ times row j from row i, add ℓ times row j to row i. (<u>NOTE</u>: This doesn't work for all L's)

So
$$\underbrace{E^{-1}\begin{bmatrix}1&0&0\\-2&1&0\\1&3&1\end{bmatrix}}_{I} A = E^{-1}\begin{bmatrix}1&2&1\\0&-1&-1\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix} U.$$

$$A = \underbrace{\begin{bmatrix}1&0&0\\2&1&0\\-1&-3&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}1&2&1\\0&-1&-1\\0&0&1\end{bmatrix}}_{U}.$$

Why LU? "cost" and accuracy - better than finding A^{-1}.

Back + Forward substitution:

<u>Example</u> Solve $\begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}3&-1\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$

First, solve $\begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$

$u = -1$, $2u + v = 2$
$v = 4$

Second solve $\begin{bmatrix}3&-1\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-1\\4\end{bmatrix}$

$y = 4$, $3x - y = -1$
$x = 1$.

IV. Transpose + Permutations.

<u>Definition</u>: If $A = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dots & \vec{v}_n\end{bmatrix}$, the <u>transpose</u> of A, denoted

$$A^T = \begin{bmatrix}\vec{v}_1\\ \vec{v}_2\\ \vdots \\ \vec{v}_n\end{bmatrix}$$

Ex: $\begin{bmatrix}1&2&3\\0&1&2\end{bmatrix}^T = \begin{bmatrix}1&0\\2&1\\3&2\end{bmatrix}$.

Facts $(AB)^{-1} = B^{-1}A^{-1}$ since $(AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I$ ✓

$(AB)^T = B^T A^T$

Why? Start with $(A\vec{x})^T = \vec{x}^T A^T$

Example $\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^T = \begin{bmatrix} 3 \\ 7 \end{bmatrix}^T = \begin{bmatrix} 3 & 7 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 7 \end{bmatrix}$. (Multiplying the same numbers).

Definition A matrix is called symmetric if $A^T = A$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ is symmetric

$\begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix}$ is symmetric.

Permutation matrices. We already saw permutations to switch 2 rows.

In general, a permutation matrix reorders any number of rows.

P = identity matrix with rows out of order

Fact: $P^{-1} = P^T$

Example When $P = P_{ij}$ switching i and j rows,

$P = P^T$ and P^TP switches the rows twice, leaving them unchanged.

So $P^TP = I$.

How do we get P?

Suppose we want row 1 to become row 3, row 3 to become row 2 and row 2 to become row 1.

Just do these things to I, and you get P.

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

When elimination requires row exchanges, $A = LU$ becomes

1) $PA = LU$ Row exchanges first

or 2) $A = LPU$ Eliminate first, then exchange rows.

Example:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 3 \\ 2 & 1 & -5 \end{bmatrix}.$$

$$PA = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 2 & 1 & -5 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$E_{31}PA = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -11 \end{bmatrix} \qquad \left(E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \right)$$

$$E_{32}(E_{31}PA) = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \qquad \left(E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \right)$$

So $$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

To get $A = L'P'U'$, eliminate first.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 3 \\ 2 & 1 & -5 \end{bmatrix} \xrightarrow{E_{32}} \underset{E_{32}A}{\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 3 \\ 0 & -5 & -11 \end{bmatrix}} \xrightarrow{E_{31}} \underset{E_{31}E_{32}A}{\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 3 \\ 0 & 0 & -1 \end{bmatrix}}.$$

$$\xrightarrow{P} \underset{PE_{31}E_{32}A}{\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}}$$

$$PE_{31}E_{32}A = U$$

$$A = (PE_{31}E_{32})^{-1}U = E_{32}^{-1}E_{31}^{-1}P^{-1}U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$