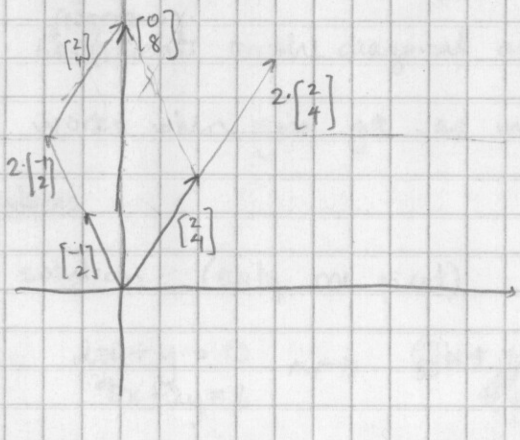
Column picture: What linear combination of $\begin{bmatrix}2\\4\end{bmatrix}$ and $\begin{bmatrix}-1\\2\end{bmatrix}$ gives $\begin{bmatrix}0\\8\end{bmatrix}$?



Class 2: 9/11/12

Outline:

I. Matrices and linear combinations

II. Matrices and systems of linear equations

III. The idea of Elimination.

IV Elimination with Matrices

Announcements

- Sign up sheet for unregistered students
- Collect HW0
- Get names

- ANY QUESTIONS?

(last time: vectors, linear combinations, dot products

Key facts: • Linear combinations of one vector form a line

of 2 vectors form a plane

of 3 vectors form 3-space

etc. generally)

• $\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \cos\theta$)

I/II. See notes from prev. class, point V.

III. Elimination

We saw that if our system of equations was triangular, we could back substitute (easy to solve).

Goal of elimination: Eliminate variables to get to a triangular system

Example: $\begin{array}{l} 2x - y = 0 \\ 4x + 2y = 8 \end{array} \rightsquigarrow \begin{array}{l} 2x - y = 0 \\ 4y = 8 \end{array}$ (Subtract 2 times row 1 from row 2)

Pivots: (Nonzero) Coefficients on the diagonal after elimination.

* Elimination works when you get as many pivots as equations.

Possible Problems

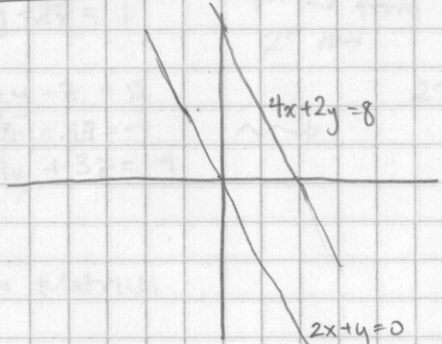
1) No solution (only one pivot)

$$\begin{array}{l} 2x + y = 0 \\ 4x + 2y = 8 \end{array} \rightsquigarrow \begin{array}{l} 2x + y = 0 \\ 0y = 8 \end{array} \longrightarrow 0 = 8 \;?!$$

Column picture: No combination of $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ give $\begin{bmatrix} 0 \\ 8 \end{bmatrix}$

(since linear combinations of $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ lie on a line $c\begin{bmatrix} 1 \\ 2 \end{bmatrix}$)

Row picture: Parallel lines don't intersect.



2) Infinitely many solutions (only one pivot)

$$\begin{array}{l} 2x + y = 4 \\ 4x + 2y = 8 \end{array} \rightsquigarrow \begin{array}{l} 2x + y = 4 \\ 0y = 0 \end{array} \qquad 0 = 0 \;\checkmark$$

Column picture: There are many ways to combine $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to get $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$, e.g. $2\begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $4\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ etc.

Row picture: The two lines coincide and any point on the line satisfies the equations.

You may have to switch rows to get to a nice triangular system.

Ex: $\begin{aligned} 0x + 2y &= 6 \\ 3x + y &= 0 \end{aligned}$ $\rightsquigarrow$ $\begin{aligned} 3x + y &= 0 \\ 2y &= 6 \end{aligned}$ (Row exchange required to get 2 pivots).

<u>Definitions</u>: A <u>singular</u> system does not have a full set of pivots.

A <u>nonsingular</u> system has a full set of pivots.

<u>Fact</u>:
- Singular systems have no solutions or infinitely many solutions.
- Nonsingular systems have exactly one solution.

A 3 by 3 example:

$\begin{aligned} 2x + y - z &= 2 \\ x + 2y \quad &= 0 \\ 2x + 5y + 2z &= 1 \end{aligned}$ $\xrightarrow[\text{1ST row from 2ND row}]{\text{subtract } 1/2}$ $\begin{aligned} 2x + y - z &= 2 \\ \tfrac{3}{2}y + \tfrac{1}{2}z &= -1 \\ 2x + 5y + 2z &= 1 \end{aligned}$

subtract row 1 from row 3 $\rightsquigarrow$ $\begin{aligned} 2x + y - z &= 2 \\ \tfrac{3}{2}y + \tfrac{1}{2}z &= -1 \\ 4y + 3z &= -1 \end{aligned}$ $\rightsquigarrow$ $\begin{aligned} 2x + y - z &= 2 \\ \tfrac{3}{2}y + \tfrac{1}{2}z &= -1 \\ \tfrac{5}{3}z &= \tfrac{5}{3} \end{aligned}$

IV. Elimination with Matrices.

<u>First</u>: Matrix multiplication

$$\begin{bmatrix} - \vec{u}_1 - \\ - \vec{u}_2 - \\ - \vec{u}_3 - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} u_1 \cdot v_1 & u_1 \cdot v_2 & u_1 \cdot v_3 \\ u_2 \cdot v_1 & u_2 \cdot v_2 & u_2 \cdot v_3 \\ u_3 \cdot v_1 & u_3 \cdot v_2 & u_3 \cdot v_3 \end{bmatrix}$$

or $$A \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ Av_1 & Av_2 & Av_3 \\ | & | & | \end{bmatrix}$$

<u>Definition</u>: An <u>elimination matrix</u> E_{ij} will subtract a multiple of row j from row i

Example: $\begin{aligned} 2x - y &= 0 \\ 4x + 2y &= 8 \end{aligned}$ Step 1 in elimination is to subtract 2 times row 1 from row 2

$E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ (Note $E_{21}\begin{bmatrix} 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$, $E_{21}\begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$)

* Elimination matrices are the identity matrix $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \ 1 \end{bmatrix}$ with a multiplier l in the ij position (i^{th} row, j^{th} column)

Example: $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ will subtract 2 times row 1 from row 2.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix}$$

* Permutation matrices are the identity matrix with rows switched.

P_{ij} switches rows i and j (and the matrix is I with rows i and j switched).

E.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ switches rows 2 and 3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Class 3: 9/13/12

Outline

I. Elimination: pathological cases
II. Elimination with matrices.
III. Matrix operations + rules.

Announcements

Collect HW 1
Registration: Stay after to fill out form.

QUESTIONS?

Last time: Triangular systems are easy to solve. Elimination: goal: make things triangular.

Key fact: Elimination works when you get as many pivots as equations.

Go back to prev. day's notes: Infinitely many solns.