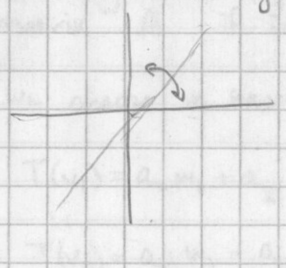
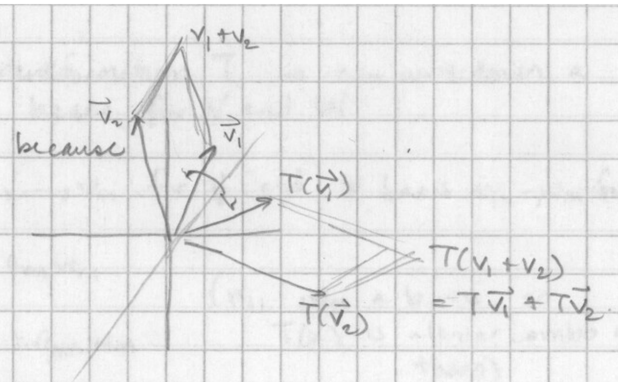
Example 2: Reflection across $y=x$



Linear because



Example 3: NOT Linear: Translation.

$$T(\vec{v}) = \vec{v}+\vec{c} \quad \left(\text{e.g. } \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}\right)$$

$$T(v_1+v_2) = \vec{v}_1+\vec{v}_2+\vec{c}, \text{ but } T(\vec{v}_1)+T(v_2) = \vec{v}_1+\vec{c}+\vec{v}_2+\vec{c}$$
$$= \vec{v}_1+\vec{v}_2+2\vec{c}$$

<u>Definition</u>: The <u>kernel</u> of a linear transformation is the set of vectors $\vec{v}\in V$ such that $T(\vec{v})=\vec{0}$.

The <u>range</u> of a linear transformation is the set of vectors $\vec{w}\in W$ such that $\vec{w}=T(\vec{v})$ for some $\vec{v}\in V$.

For rotation and reflection, kernel $=\{\vec{0}\}$.
range $=\mathbb{R}^2$.

Example: Fix a vector $\vec{a}\in\mathbb{R}^2$, let $V=\mathbb{R}^2$, $W=\mathbb{R}$
Let $T(\vec{x})=\vec{a}\cdot\vec{x}$

This is linear. $\left(\vec{a}\cdot(\vec{x}+\vec{y}) = \vec{a}\cdot\vec{x}+\vec{a}\cdot\vec{y} = T(\vec{x})+T(\vec{y})\right.$
and $\left.T(c\vec{x}) = \vec{a}\cdot(c\vec{x}) = c(\vec{a}\cdot\vec{x}) = cT(\vec{x})\right)$

Kernel is the vectors perpendicular to $\vec{a}$.
Range $=\mathbb{R}$

Class 17: 11/6

Outline
I. Review Lin Trans.
II. Matrix of a lin. Trans.

Midterm 2: 7-8:30pm Nov. 13 Salomon 001
Homework: Don't miss the handout (with notes at the end)

Review: A linear transformation is a map $T: V\longrightarrow W$ satisfying
1) $T(v_1+v_2) = T(v_1)+T(v_2)$
2) $T(cv_1) = cT(v_1)$

Examples: Rotation, reflection (of $\mathbb{R}^2$), dot product (above)

Review def. of kernel and range

Key idea of today: For any linear transformation T, we can writedown a matrix A if we choose bases for V and W.

Say we choose a basis $v_1, v_2, \dots, v_n$ for V and a basis $w_1, \dots, w_m$ for W.

Then $T(v_1) = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m$

$\vdots$

$T(v_n) = a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m$

($w_1, \dots, w_m$ a basis, so $T(v_i)$ is a linear combo of them)

We write the corresponding matrix A.

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} \\ \begin{array}{ccc} \uparrow & & \uparrow \\ T(v_1) & \cdots & T(v_n) \end{array} \end{array} \begin{array}{l} \leftarrow w_1 \\ \leftarrow w_2 \\ \\ \leftarrow w_m \\ \\ \\ \end{array}$$

Then if $v \in V$ and $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$, we can find $T(v)$ using A.

We've set up a correspondence

$$V \longleftrightarrow \mathbb{R}^n \qquad\qquad W \longleftrightarrow \mathbb{R}^m$$

$$c_1 v_1 + \dots + c_n v_n \longleftrightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \qquad d_1 w_1 + \dots + d_m w_m \longleftrightarrow \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}$$

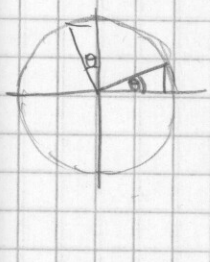
So $T(v) \longleftrightarrow A \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

<u>Great news</u>: We can use A to find out a lot about T

- range $\longleftrightarrow$ column space
- kernel $\longleftrightarrow$ nullspace
- eigenvalues & eigenvectors (if $T: V \to V$)
- invertibility

<u>Example</u>: The matrix for rotation by θ in $\mathbb{R}^2$

Choose the standard basis for $\mathbb{R}^2$: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



$$T(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \cos\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\vec{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = -\sin\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

From the geometry: No fixed directions, so no real eigenvalues.
The matrix backs this up.

Also, T^2 = rotation by 2θ has matrix A^2. (check trig rules).

(Fact: Composing transformations corresponds to multiplying matrices).

Example: V = 2x2 matrices, W = 1x2 matrices.

$T(C) = [1\ 2]C$. Find the matrix for T.

First choose bases for V and W.

For V: $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

For W: $F_1 = [1\ 0]$ $F_2 = [0\ 1]$.

Now compute $T(E_i)$ for each i.

$$T(E_1) = [1\ 2]\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = [1\ 0] = F_1$$

$$T(E_2) = [1\ 2]\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = [0\ 1] = F_2$$

$$T(E_3) = [1\ 2]\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = [2\ 0] = 2F_1$$

$$T(E_4) = [1\ 2]\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = [0\ 2] = 2F_2$$

So $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$.

What is dim(kernel T)? $\dim \ker T = \dim N(A) = 2$.

Find a basis for $\ker T$.

Basis for $N(A)$: $S_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $S_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

So basis for $\ker T$: $S_1 = -2E_1 + E_3 = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}$

$S_2 = -2E_2 + E_4 = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$

Example: $V = W$ = space of polynomials of degree ≤ 2.
$T(f) = (1+t) f'(t)$.
Matrix?
Choose a basis for V: $1, t, t^2$

$$\left.\begin{array}{l} T(1) = 0 = 0\cdot 1 + 0\cdot t + 0\cdot t^2 \\ T(t) = 1+t = 1\cdot 1 + 1\cdot t + 0\cdot t^2 \\ T(t^2) = (1+t)(2t) = 0\cdot 1 + 2\cdot t + 2\cdot t^2 \end{array}\right\} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Q: Find g so that g is not in the range of T.

What does this question mean for A?
Find a vector not in the column space.

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & a \\ 0 & 1 & 2 & b \\ 0 & 0 & 2 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & 0 & a \\ 0 & 0 & 2 & b-a \\ 0 & 0 & 2 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & 0 & a \\ 0 & 0 & 2 & b-a \\ 0 & 0 & 0 & c-b+a \end{array}\right].$$

So we need $c-b+a=0$ for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to be in $C(A)$

So $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in $C(A)$, and $f(t)=1$ is not in the range of T.

Conclusion: Our matrix experience lets us solve lots of linear transformation questions.

"Change of basis" What happens if we had chosen different bases?
* We get a different matrix, even though the transformation is the same.

Example: Projection onto the line spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$.

Choice 1: Standard basis
$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$$A_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Choice 2: Basis of eigenvectors
$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 0$

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Key fact: Different bases give different matrices, but the matrices will be similar.

In our example, $A_1 = S A_2 S^{-1}$.