Class 15: 10/25

Outline: I. Symmetric Matrices
II. Positive Definite Matrices

Questions from last time? (Systems of linear differential equations).

I. Symmetric matrices ($A^T = A$).

Key facts about symmetric matrices:

1. A symmetric matrix has all real eigenvalues
2. Eigenvectors can be taken to be orthonormal
3. Symmetric matrices can always be diagonalized.

Spectral Theorem: Every symmetric matrix A can be written as

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^T \quad (Q^{-1} = Q^T)$$

Example 1: Last time we had P = projection onto $y=x$ in $\mathbb{R}^2$.
Projection matrices are symmetric.
- We had orthogonal eigenvectors (can be normalized).
- Eigenvalues were 0 and 1 (real numbers).

Example 2: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - 4 = -5\lambda + \lambda^2$$

Eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 5$

Eigenvectors For λ_1: Solve $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \vec{x} = \vec{0} \rightsquigarrow \vec{x}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

For λ_2: Solve $\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \vec{x} = \vec{0} \rightsquigarrow \vec{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(Note: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a basis for $N(A)$.

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a basis for the row space / column space.

These are orthogonal!).

Eigenvectors aren't orthonormal yet.

How do we make them orthonormal?
Normalize: In this case, divide by $\sqrt{5}$.

So $Q = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$

$$A = Q\Lambda Q^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 0 & 5 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \checkmark$$

Why are eigenvalues real?

$$A\vec{x} = \lambda\vec{x} \implies A\overline{\vec{x}} = \overline{\lambda}\,\overline{\vec{x}}, \text{ and } \overline{\vec{x}}^T A = \overline{\vec{x}}^T\overline{\lambda}$$

$$\text{So } \overline{\vec{x}}^T A\vec{x} = \overline{\vec{x}}^T\lambda\vec{x} = \overline{\vec{x}}^T\,\overline{\lambda}\vec{x} \qquad \overline{\vec{x}}^T\vec{x} = |x_1|^2 + \cdots + |x_n|^2 \neq 0$$

So $\lambda = \overline{\lambda}$ and λ is real.

Why are eigenvectors orthogonal when they have different eigenvalues?

Say $A\vec{x}_1 = \lambda_1\vec{x}_1$ and $Ax_2 = \lambda_2\vec{x}_2$.

$$x_1^T A x_2 = \vec{x}_1^T\lambda_2\vec{x}_2 = \lambda_2\vec{x}_1^T\vec{x}_2$$

$$x_1^T A x_2 = x_1^T A^T x_2 = (Ax_1)^T x_2 = \lambda_1\vec{x}_1^T\vec{x}_2$$

Since $\lambda_1 \neq \lambda_2$, $\vec{x}_1^T\vec{x}_2 = 0$ and $\vec{x}_1$ and $\vec{x}_2$ are orthogonal

Fact. $A = Q\Lambda Q^T$ implies that

$A = \lambda_1 P_1 + \cdots + \lambda_n P_n$ where P_i is projection onto the eigenspace with eigenvalue λ_i.

(i.e. $A = \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \cdots$)

Fact: If $A \to U$ requires no permutations, then the signs of the pivots are the same as the signs of the eigenvalues.

Example $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ Pivots? $U = \begin{bmatrix} 1 & 3 \\ 0 & -8 \end{bmatrix}$ 1, −8

Eigenvalues? $A - \lambda I = \begin{bmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix}$

Characteristic poly $= (1-\lambda)^2 - 9 = \lambda^2 - 2\lambda - 8$

$\lambda_1 = 4,\ \lambda_2 = -2$

If A is not symmetric, the signs might not match.

$$A = \begin{bmatrix} 1 & 0 \\ -1 & -4 \end{bmatrix} \quad \text{Eigenvalues } -1, -2 \quad \text{Pivots: } 1, 2.$$

II. Positive Definite Matrices:

Definition: A symmetric matrix with all positive eigenvalues is called positive definite.

For 2x2: When is $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ positive definite?

Need λ_1 and $\lambda_2 > 0$, so $\lambda_1 \lambda_2 = \text{determinant} > 0$
and $\lambda_1 + \lambda_2 = \text{trace} > 0$.

So a symmetric 2x2 matrix is positive definite when
$ac - b^2 > 0$
$a + c > 0$.

But if a or $c < 0$, $ac - b^2 < 0$, so really need $a > 0$. and $c > 0$.

2x2 test: $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite when
$ac - b^2 > 0$
$a > 0$.

Alternate definition: A symmetric matrix A is positive definite if $\vec{x}^T A \vec{x} > 0$ for every $\vec{x} \neq \vec{0}$.

Need $[x, y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [x, y] \begin{bmatrix} ax + by \\ bx + cy \end{bmatrix}$

$= ax^2 + 2bxy + cy^2. > 0.$

Facts: 1) If A and B are positive definite, so is $A + B$.
Use alternate def.
$\vec{x}^T (A+B) \vec{x} = \vec{x}^T A \vec{x} + \vec{x}^T B \vec{x} > 0$
and $(A+B)^T = A^T + B^T = A + B$ so symmetric.

2) If $A = R^T R$, A is symmetric. If R has independent columns, $A = R^T R$ is positive definite.
$\vec{x}^T A \vec{x} = \vec{x}^T R^T R \vec{x} = (R\vec{x})^T R\vec{x} = \|R\vec{x}\|^2 > 0$ if R has indep. columns

Equivalent properties for symmetric matrices

1. Pivots are all positive and no permutations were necessary to get to U.
2. All n upper left determinants are positive
3. All n eigenvalues are positive
4. $\vec{x}^T A \vec{x} > 0$ except for $\vec{x} = 0$
5. $A = R^T R$ for a matrix R with independent columns.

(If a symmetric matrix has any one of these properties, it is positive definite).

Example: $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Test 2: determinants are $2, 3, 4$.

Test 3: eigenvalues? $2,\ 2+\sqrt{2},\ 2-\sqrt{2}$

Test 1: pivots: $2, \frac{3}{2}, \frac{4}{3}$.

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$ (Test 5)

Test 4: $\vec{x}^T A \vec{x} = 2\left(x_1^2 - x_1 x_2 + x_2^2 - x_2 x_3 + x_3^2\right)$

$$= 2\left(x_1 - \tfrac{1}{2}x_2\right)^2 + \tfrac{3}{2}\left(x_2 - \tfrac{2}{3}x_3\right)^2 + \tfrac{4}{3}x_3^2 > 0$$