Class 14 : 10/23

Outline
I. Eigenvalues & Eigenvectors & Differential equations.
II. Matrix exponential

Recall from calculus: $\frac{du}{dt} = \lambda u$ has solution $u(t) = Ce^{\lambda t}$.

Now, we want to solve systems of linear differential equations.

Example: $\frac{dy}{dt} = z$ $\rightsquigarrow$ $\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$

$\frac{dz}{dt} = y$ $\qquad \frac{d}{dt}\vec{u} = A\vec{u}$

Use eigenvalues & eigenvectors!

A has eigenvalues ± 1 and eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(How do we find these?

$$\text{char poly} = \det\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

So $\lambda_1 = 1$, $\lambda_2 = -1$

$\vec{x}_1$ solves $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\vec{x}_1 = 0 \rightsquigarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x}_2$ solves $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\vec{x}_2 = 0 \rightsquigarrow \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So if $\vec{u}_1(t) = e^{\lambda_1 t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\frac{du_1}{dt} = \lambda_1 e^{\lambda_1 t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = A\vec{u}_1(t)$

$\vec{u}_2(t) = e^{\lambda_2 t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \frac{du_2}{dt} = \lambda_2 e^{\lambda_2 t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = A\vec{u}_2(t)$.

So complete solution:

$$u(t) = Ce^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} Ce^{t} + De^{-t} \\ Ce^{t} - De^{-t} \end{bmatrix}$$

Use this to get long-term behavior:

As $t \to \infty$, $u(t) \to \begin{bmatrix} Ce^{t} \\ Ce^{t} \end{bmatrix}$

General algorithm:

Solving: $\frac{d}{dt}\vec{u} = A\vec{u}$

1. Find eigenvalues & eigenvectors for A
   $\lambda_1, \dots, \lambda_n$ $\quad \vec{x}_1, \dots, \vec{x}_n$
2. Solution is $u(t) = c_1 e^{\lambda_1 t}\vec{x}_1 + \dots + c_n e^{\lambda_n t}\vec{x}_n$.
3. If necessary, use initial conditions to solve for $c_1, \dots, c_n$.

(Note: For this to work, we need a basis of eigenvectors)

Second order equations

Say we want to solve $\quad my'' + by' + ky = 0$.

Use previous: Set $\vec{u} = \begin{bmatrix} y \\ y' \end{bmatrix}$

Equation says $\quad \frac{dy}{dt} = y'$

$$\frac{dy'}{dt} = y'' = -\frac{b}{m}y' - \frac{k}{m}y$$

$$\frac{d\vec{u}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}$$

For simplicity, let's say $m=1$.

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -k & -b-\lambda \end{bmatrix} = \lambda^2 + b\lambda + k$$

Eigenvalues are roots of $\lambda^2 + b\lambda + k = 0$
Solve this the same way as before.

Example: Solve $\quad y'' + y' - 2y = 0$.

Set $\vec{u} = \begin{bmatrix} y \\ y' \end{bmatrix}$ $\quad$ Then $\frac{d}{dt}\vec{u} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$

Eigenvalues + evectors: $\quad$ char poly $= \det\begin{bmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix} = \lambda^2 + \lambda - 2$

$$= (\lambda+2)(\lambda-1)$$

$\lambda_1 = -2$, $\quad \lambda_2 = 1$

For $\vec{x}_1$: $\quad \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\vec{x} = 0 \longrightarrow \vec{x}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\vec{x}_2$: $\quad \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\vec{x} = 0 \longrightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} -Ce^{-2t} + De^{t} \\ 2Ce^{-2t} + De^{t} \end{bmatrix}$

Matrix exponentials

$$\frac{du}{dt} = \lambda u \quad \text{has solution} \quad u = Ce^{\lambda t}$$

$$\frac{d\vec{u}}{dt} = A\vec{u} \quad \text{should have solution} \quad \vec{u}(t) = Ce^{At}\vec{u}(0).$$

What does e^{At} mean?!

Recall: $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

So $e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$

Use diagonalization to make these matrix powers easy

$$A = S\Lambda S^{-1}, \text{ so } (At)^k = A^k t^k = (S\Lambda S^{-1})^k t^k$$
$$= \underbrace{(S\Lambda S^{-1})(S\Lambda S^{-1})\cdots(S\Lambda S^{-1})}_{k} t^k$$
$$= S\Lambda^k S^{-1} t^k = S(\Lambda^k t^k)S^{-1}$$

So: $e^{At} = S e^{\Lambda t} S^{-1}$, and $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$

Advantage: Matrix exponential works even when we're missing an eigenvector

Example: Solve $y'' - 2y' + y = 0 \rightsquigarrow \frac{d\vec{u}}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$

Characteristic polynomial is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$.

eigenvectors: Solve $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\vec{x} = 0 \rightsquigarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Only one eigenvector.

But: use matrix exponential to solve.

$$e^{At} = e^{tI}e^{(A-I)t}$$
$$= e^t\left(I + (A-I)t + \frac{((A-I)t)^2}{2} + \dots\right)$$

But $(A-I)^2 = 0$, so we get

$$e^{At} = e^t(I + (A-I)t)$$

$$u(t) = e^{At}\vec{u}(0) = e^t\left[I + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}t\right]\vec{u}(0) = e^t y(0) - te^t y(0) + te^t y'(0)$$