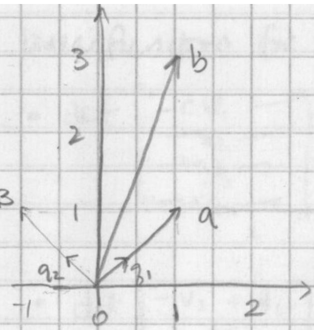
Example 2×2 example

$\vec{a}=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b}=\begin{bmatrix}1\\3\end{bmatrix}$



$$A=\begin{bmatrix}1\\1\end{bmatrix}, \quad B=\begin{bmatrix}1\\3\end{bmatrix}-\frac{a^Tb}{a^Ta}a=\begin{bmatrix}1\\3\end{bmatrix}-\frac{4}{2}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}-\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$$

Normalize: $q_1=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$ $\quad q_2=\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

Class 11: 10/11

Collect HW

Outline

I. Determinants
- Properties
- 3 methods of computing determinants

What is determinant? A number associated to a square matrix.
Another measure of size.

(Rank: How many equations do you really have / dim of $C(A)$
Determinant: How much does your matrix stretch vectors)

Example: $\det(I)=1$ no stretch

$$\det\begin{bmatrix}4&0\\0&4\end{bmatrix}=16 \quad \left(\begin{bmatrix}4&0\\0&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4x\\4y\end{bmatrix}\right)$$

$$\det\begin{bmatrix}4&0\\0&1\end{bmatrix}=4 \quad \left(\begin{bmatrix}4&0\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4x\\y\end{bmatrix}\right)$$

Properties of Determinant

1) $\det(I)=1$

2) Switching 2 rows changes the sign of the determinant

$$\begin{vmatrix}a&b\\c&d\end{vmatrix} \text{ vs } \begin{vmatrix}c&d\\a&b\end{vmatrix}$$

3) Determinant is a linear function for each row (separately).

This means: • $\det \begin{bmatrix} — cv_1 — \\ — v_2 — \\ — v_3 — \\ \vdots \\ — v_m — \end{bmatrix} = c \det \begin{bmatrix} — v_1 — \\ \vdots \\ — v_m — \end{bmatrix}$

• $\det \begin{bmatrix} — v_1 + w_1 — \\ — v_2 — \\ \vdots \\ — v_m — \end{bmatrix} = \det \begin{bmatrix} — v_1 — \\ — v_2 — \\ \vdots \\ — v_m — \end{bmatrix} + \det \begin{bmatrix} — w_1 — \\ — v_2 — \\ \vdots \\ — v_m — \end{bmatrix}$.

4) Adding rows to each other doesn't change determinant

$$\det \begin{bmatrix} — v_1 — \\ \\ — v_m — \end{bmatrix} = \det \begin{bmatrix} — \vec{v}_1 + l\vec{v}_2 — \\ — \vec{v}_2 — \\ \\ — \vec{v}_m — \end{bmatrix}$$

(Elimination does not change determinant.
$\det A = \det U$)

5) $\det(AB) = \det(A)\det(B)$

6) If A is triangular, $\det(A) =$ product of diagonal entries

7) If A has 2 rows equal to each other or a row of 0's, then $\det A = 0$.

8) $\det(A) = \det(A^T)$

$\hookrightarrow$ <u>NOTE</u>: This means all properties hold when you replace "row" by "column".

e.g. If A has a column of 0's, $\det A = 0$.

<u>Fact</u>: If $\det A = 0$, A is singular. If $\det A \neq 0$, A is invertible

We can use the properties to compute some determinants

$$\begin{vmatrix} 1 & 6 & 5 \\ 0 & 0 & 2 \\ 0 & 3 & -1 \end{vmatrix} = \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 6 & 5 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}\right) = (-1)\cdot 1\cdot 3\cdot 2 = -6.$$

More generally, there are 3 ways to compute determinants.

1) Pivot formula
2) Permutations ("Big Formula")
3) Cofactors.

1) Pivot formula:

Factor $PA = LU$

Then $\det A = \dfrac{\det(U)}{\det(P)} = \pm \det(U)$

(See example above).

2) Permutations: There are $n!$ terms that we sum

$$\det A = \sum \det(P)\, a_{1\alpha_1} a_{2\alpha_2} \cdots a_{n\alpha_n}$$

where P is the permutation that makes the α_ith row the ith row.

Example:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

27 terms, but many are 0. The nonzero ones have no zero rows or columns. There are $6 = 3!$ of these

$$\begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

So $\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} + (-1)(a_{11}a_{23}a_{32})$
$- a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

Example: $\det \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 1$ (P switches row 1 and 2 and row 3 and 4 $\rightsquigarrow$ 2 row switches)

3) Cofactors

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

where $C_{ij} = (-1)^{i+j} \det M_{ij}$

<u>Example</u>: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$+ \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$

$= a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$

So: choose a row (or column) to expand along.
For each entry in the row, cross out its row and column to get an $(n-1) \times (n-1)$ submatrix.

Take the submatrix's determinant.

Example: $\det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = 2 \cdot \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$

$= 2\left(2 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}\right) + (-1)\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 2(6 + -2) + (-3) = 5$