Practice Problems for Midterm 1 Solutions

1) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 3 & 6 & 3 \end{bmatrix}$.

To compute A^{-1}, augment by I and row reduce.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 3 & 6 & 3 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 6 & 6 & -3 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 3 & 0 & -3 & 3 & -1 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 1 & 0 & -1 & 1 & -1/3 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right].$$

$$A^{-1} = \begin{bmatrix} 3/2 & -1 & 1/2 \\ -1 & 1 & -1/3 \\ 1/2 & -1 & 1/2 \end{bmatrix}$$

(Check: $AA^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 3 & 6 & 3 \end{bmatrix}\begin{bmatrix} 3/2 & -1 & 1/2 \\ -1 & 1 & -1/3 \\ 1/2 & -1 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓)

2) $\vec{a} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 1 \end{bmatrix}$.

$\vec{A} = \vec{a}$.

$$\vec{B} = \vec{b} - \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\vec{a} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{C} = \vec{c} - \frac{\vec{a}^T\vec{c}}{\vec{a}^T\vec{a}}\vec{a} - \frac{\vec{B}^T\vec{c}}{\vec{B}^T\vec{B}}\vec{B} = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 1 \end{bmatrix} - \frac{-4}{4}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} - \frac{-4}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

So $\vec{q}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, $\vec{q}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{q}_3 = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ is an orthonormal basis for the span of $\vec{a}, \vec{b}$ and $\vec{c}$.

2) cont.

So $Q = \begin{bmatrix} 1/2 & 1/2 & 2/\sqrt{10} \\ 1/2 & -1/2 & 1/\sqrt{10} \\ -1/2 & -1/2 & 2/\sqrt{10} \\ -1/2 & 1/2 & 1/\sqrt{10} \end{bmatrix}$

and $R = Q^T A = \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 2/\sqrt{10} & 1/\sqrt{10} & 2/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ -1 & -2 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 10/\sqrt{10} \end{bmatrix}$

So $A = \begin{bmatrix} 1/2 & 1/2 & 2/\sqrt{10} \\ 1/2 & -1/2 & 1/\sqrt{10} \\ -1/2 & -1/2 & 2/\sqrt{10} \\ -1/2 & 1/2 & 1/\sqrt{10} \end{bmatrix} \begin{bmatrix} 2 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & \sqrt{10} \end{bmatrix}$

3) For what values of C does $\begin{bmatrix} 3 & 1 & 6 \\ 2 & 0 & 4 \\ 1 & 2 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ c \\ c \end{bmatrix}$ have a solution?

$$\left[\begin{array}{ccc|c} 3 & 1 & 6 & 1 \\ 2 & 0 & 4 & c \\ 1 & 2 & 2 & c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & c \\ 2 & 0 & 4 & c \\ 3 & 1 & 6 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & c \\ 0 & -4 & 0 & -c \\ 0 & -5 & 0 & 1-3c \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & c \\ 0 & -4 & 0 & -c \\ 0 & 0 & 0 & 1-3c+\frac{5c}{4} \end{array}\right]$$

This has solutions when $1 - \frac{7}{4}c = 0$, i.e. when $c = \frac{4}{7}$

4) The space of symmetric matrices has basis

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and is 3 dimensional.

(This is a basis because a symmetric matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

and they are independent).

5) $\vec{v}_1 = (1, 2, 0, 1)$ $\vec{v}_2 = (1, 1, 1, 1)$, $\vec{v}_3 = (2, 3, 1, 0)$.

a) If $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 0 \end{bmatrix}$, a vector perpendicular to $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a vector in the nullspace of A.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$\uparrow$ free variable

$S_1 = \begin{bmatrix} x_1 \\ x_2 \\ 1 \\ x_3 \end{bmatrix}$: $\left.\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ -x_2 + 1 &= 0 \\ -2x_3 &= 0 \end{aligned}\right\} \rightarrow \begin{aligned} x_1 &= -2 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$

So $\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ is perpendicular to $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

(check: $\vec{v}_1 \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 0$? $\vec{v}_2 \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 0$? $\vec{v}_3 \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 0$!)

b) $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & -1 \\ 0 & 1 & 1 & c \\ 1 & 1 & 0 & 0 \end{bmatrix}$ is invertible if it has independent columns and 4 pivots (full rank).

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & -1 \\ 0 & 1 & 1 & c \\ 1 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & c \\ 0 & 0 & -2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & c-3 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

For there to be 4 pivots, we need $c - 3 \neq 0$.

So any $c \neq 3$ will make the matrix invertible.

6) A a 5×4 matrix

a) $A\vec{v}_1$, $A\vec{v}_2$ and $A\vec{v}_3$ being independent in $\mathbb{R}^5$ means that if

$$c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + c_3 A\vec{v}_3 = 0, \quad \text{then } c_1 = c_2 = c_3 = 0.$$

b) Say $c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + c_3 A\vec{v}_3 = 0$.

Then $A(c_1\vec{v}_1) + A(c_2\vec{v}_2) + A(c_3\vec{v}_3) = 0$

and $A(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = 0$.

So $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ is in $N(A)$.

But $N(A) = 0$. So $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$.

But $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are independent, implying that $c_1 = c_2 = c_3 = 0$.

Thus, whenever $c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + c_3 A\vec{v}_3 = 0$, $c_1 = c_2 = c_3 = 0$ and $A\vec{v}_1$, $A\vec{v}_2$ and $A\vec{v}_3$ are independent.