Practice Problem Solutions

1) A a 3x3 matrix and $\det(A) = 2$.

a) $\det(3A) = 3^3 \det(A) = 54$

b) $\det(-A) = (-1)^3 \det(A) = -2$

c) $\det(A^2) = \det(A)^2 = 4$

d) $\det(A^T) = \det(A) = 2$

e) $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{2}$

2)

$H_{k+2} = 2H_k + H_{k+1}$

$H_{k+1} = H_{k+1}$

If $u_k = \begin{bmatrix} u_{k+1} \\ u_k \end{bmatrix}$, then $u_{k+1} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} u_k$

Eigenvalues: $\det \begin{bmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$

$\lambda_1 = 2$

$\lambda_2 = -1$

Eigenvectors: $\lambda_1 = 2$: $\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \vec{x} = 0$: $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$: $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \vec{x} = 0$; $\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. To find u_{100}, use $A^{100} u_0$:

$u_0 = \frac{1}{3} \left[\vec{x}_1 + \vec{x}_2 \right]$.

So $A^{100} u_0 = 2^{100} \dfrac{\vec{x}_1}{3} + \dfrac{1^{100} \vec{x}_2}{3}$

$= \dfrac{1}{3} \begin{bmatrix} 2^{101} + 1 \\ 2^{100} - 1 \end{bmatrix} = \begin{bmatrix} H_{101} \\ H_{100} \end{bmatrix}$

a) $H_{100} = \dfrac{2^{100} - 1}{3}$

b) $\lim\limits_{k \to \infty} \dfrac{H_{k+1}}{H_k} = 2$ $\left(\text{it's } \lim\limits_{k\to\infty} \dfrac{\left(\frac{2^{k+1}+1}{3}\right)}{\left(\frac{2^k - 1}{3}\right)} = 2 \right)$

3)

$$p_1'(t) = 7w_1 - 2s_2$$
$$p_2'(t) = 7w_2 - 2s_1$$

a) $w_1 = \frac{1}{2}p_1$ and $s_1 = \frac{1}{2}p_1$
$w_2 = 0$ and $s_2 = p_2$

$\Big\}$ So we get $p_1'(t) = \frac{7}{2}p_1 - 2p_2$
$p_2'(t) = -\cancel{p}\,p_1$

set $u(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$ Then $\frac{d\vec{u}}{dt} = \begin{bmatrix} \frac{7}{2} & -2 \\ -1 & 0 \end{bmatrix} \vec{u}$.

b) Eigenvalues. $\det \begin{bmatrix} 7/2 - \lambda & -2 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 - \frac{7\lambda}{2} - 2 = 0$

$$\lambda = \frac{\frac{7}{2} \pm \sqrt{\frac{49}{4} + 8}}{2} = \frac{\frac{7}{2} \pm \frac{9}{2}}{2}$$

$\lambda_1 = 4, \quad \lambda_2 = -\frac{1}{2}$

Eigenvectors. $\lambda_1 = 4$: $\begin{bmatrix} -\frac{1}{2} & -2 \\ -1 & -4 \end{bmatrix} \vec{x} = 0 \rightsquigarrow \vec{x}_1 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$\lambda_2 = -\frac{1}{2}$: $\begin{bmatrix} 4 & -2 \\ -1 & \frac{1}{2} \end{bmatrix} \vec{x} = 0 \rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$u(t) = C e^{4t} \begin{bmatrix} -4 \\ 1 \end{bmatrix} + D e^{-\frac{t}{2}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$u(0) = \begin{bmatrix} 18 \\ 18 \end{bmatrix}$, so $\left.\begin{aligned} -4C + D &= 18 \\ C + 2D &= 18 \end{aligned}\right\} \rightarrow$

$$\begin{aligned} -8C + 2D &= 36 \\ C + 2D &= 18 \\ \hline -9C &= 18 \\ C &= -2 \\ D &= 10 \end{aligned}$$

So
$$p_1(t) = 8e^{4t} + 10e^{-t/2}$$
$$p_2(t) = -2e^{4t} + 20e^{-t/2}$$

c) Player 1 wins. (His population goes to infinity, while player 2's population decreases and will eventually be 0).

4) $A = \begin{bmatrix} 1 & a \\ a & 2 \end{bmatrix}$.

a) A is positive definite if the 2 upper left determinants are positive.

$$\det [1] = 1 \quad \checkmark$$

$$\det \begin{bmatrix} 1 & a \\ a & 2 \end{bmatrix} = 2 - a^2 > 0 \quad \rightsquigarrow \quad 2 > a^2$$

so if $-\sqrt{2} < a < \sqrt{2}$, A is positive definite.

b) If $a = -1$, then $-\sqrt{2} < a < \sqrt{2}$, so A is positive definite.

c)

$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ eigenvalues: $\det \begin{bmatrix} 1-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 3\lambda + 2 - 1 = \lambda^2 - 3\lambda + 1$

$$\lambda = \frac{3 \pm \sqrt{9-4}}{2}$$

$$\lambda_1 = \frac{3+\sqrt{5}}{2}, \quad \lambda_2 = \frac{3-\sqrt{5}}{2}$$

eigenvectors:

For λ_1: $\begin{bmatrix} 1-\lambda_1 & -1 \\ -1 & 2-\lambda_1 \end{bmatrix} \vec{x} = 0$

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1-\lambda_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{bmatrix}$$

For λ_2: $\begin{bmatrix} 1-\lambda_2 & -1 \\ -1 & 2-\lambda_2 \end{bmatrix} \vec{x}_2 = 0$

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 1-\lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix}$$

If $\Lambda = \begin{bmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{bmatrix}$ $\quad Q = \begin{bmatrix} \frac{1}{\|x_1\|} & \frac{1}{\|x_2\|} \\ \frac{1-\sqrt{5}}{2\|x_1\|} & \frac{-1+\sqrt{5}}{2\|x_2\|} \end{bmatrix}$

then $A = Q \Lambda Q^T$

d) $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

So $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

5) a) $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Eigenvalues: $\det \begin{bmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda)(2-\lambda)$

So $\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 2$.

b) Note that corresponding to $\lambda_1 = \lambda_2 = 1$, we have only one independent eigenvector:

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x} = 0 \quad \rightsquigarrow \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

So $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, which has the same eigenvalues but 2 independent eigenvectors with eigenvalue 1, is not similar to A.

c) $C = MAM^{-1}$ is similar to A. Choose a simple M, like $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Then $M^{-1} = M$.

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

6) $T(M) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} M \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$.

a) $T(D_1) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = 4 D_3$

$T(D_2) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} = 4 D_2$

$T(D_3) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = 4 D_1$

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

6) b) Eigenvalues:

$$\det \begin{bmatrix} -\lambda & 0 & 4 \\ 0 & 4-\lambda & 0 \\ 4 & 0 & -\lambda \end{bmatrix} = \lambda^2(4-\lambda) - 16(4-\lambda)$$

$$= (\lambda^2-16)(4-\lambda) = (\lambda+4)(\lambda-4)(4-\lambda).$$

So $\lambda_1 = -4$

$\lambda_2 = \lambda_3 = 4.$

Eigenvectors: $\lambda_1 = -4$: $\begin{bmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{bmatrix} \vec{x} = 0 \rightsquigarrow \vec{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ corresponds to $-D_1 + D_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\lambda_2 = \lambda_3 = 4$: $\begin{bmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{bmatrix} \vec{x} = 0 \rightsquigarrow \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\vec{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

These correspond to $D_1 + D_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $D_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

So eigenvalues are $\lambda_1 = -4$ with eigenvector $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda_2 = \lambda_3 = 4$ with eigenvectors $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

c) T is diagonalizable. We have the full number of eigenvectors.

d) $T^{-1}(M) = BMB$.

$T^{-1}(T(M)) = M$ on the one hand. But also

$$T^{-1}(T(M)) = T^{-1}\left(\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} M \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \right)$$

$$= B \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} M \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} B.$$

So B should be $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

e) The matrix for T^{-1} is A^{-1}, which is

$$\begin{bmatrix} 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 \end{bmatrix}.$$

f) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} M \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$ translates to solving

$$A\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 4 & 1 \\ 0 & 4 & 0 & -1 \\ 4 & 0 & 0 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 4 & 0 & 0 & 3 \\ 0 & 4 & 0 & -1 \\ 0 & 0 & 4 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/4 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 1/4 \end{array}\right]$$

So $M = \begin{bmatrix} 3/4 & -1/4 \\ -1/4 & 1/4 \end{bmatrix}$ has $T(M) = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$.