

MATH 520 - PRACTICE PROBLEMS FOR MIDTERM II

FOR FULL CREDIT, SHOW ALL WORK
NO CALCULATORS

1. Suppose A is a 3-by-3 matrix with determinant 2. Compute

- (a) $\det(3A)$
- (b) $\det(-A)$
- (c) $\det(A^2)$
- (d) $\det(A^T)$
- (e) $\det(A^{-1})$

2. Define the Hibbonacci numbers by the recursion relation

$$H_{k+2} = 2H_k + H_{k+1}$$

and initial conditions $H_0 = 0$, $H_1 = 1$.

- (a) What is H_{100} ?
 - (b) What is the limit $\lim_{k \rightarrow \infty} \frac{H_{k+1}}{H_k}$?
3. In a two-player strategy game, each player starts with 18 pieces which they can designate as “workers” or “soldiers.” Before the game, each player decides what proportion of their population will be workers (and the remaining pieces will be soldiers). This proportion will be fixed for the whole game. A player’s workers help to increase their population, while a player’s soldiers decrease the other player’s population (in short, the pieces either make love or war). The differential equations that model this game are

$$\begin{aligned} p_1'(t) &= 7w_1 - 2s_2 \\ p_2'(t) &= 7w_2 - 2s_1 \end{aligned}$$

where p_i is the i -th player’s total population, w_i is the number of workers and s_i is the number of soldiers.

Player 1 chooses a balanced approach of half workers and half soldiers, while Player 2, hoping to quickly decimate his opponent, decides to devote his entire population to soldiers.

- (a) Rewrite the differential equations above as a system of equations just in terms of p_1 and p_2 .
- (b) Find equations for $p_1(t)$ and $p_2(t)$.

- (c) The game ends when a player has no more pieces, or if the players agree that one of their populations is growing at a faster rate than the other's. Who wins this game?

4. Let

$$A = \begin{bmatrix} 1 & a \\ a & 2 \end{bmatrix}.$$

- (a) For which values of a is A positive definite?
- (b) Show A is positive definite for $a = -1$.
- (c) Write $A = Q\Lambda Q^T$ when $a = -1$.
- (d) Write $A = R^T R$ for some matrix R when $a = -1$.

5. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$

- (a) What are the eigenvalues of A ?
- (b) Give an example of a matrix B with the same eigenvalues as A that is not similar to A .
- (c) Give an example of a matrix $C \neq A$ such that C is similar to A .

6. The vector space V of all 2×2 symmetric matrices has a standard basis of $D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, D_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$

If M is a matrix in the vector space V , let $T : V \rightarrow V$ be the linear transformation given by $T(M) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} M \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$

- (a) Find the matrix A associated to T .
- (b) What are the eigenvalues and eigenvectors of T ?
- (c) Is T diagonalizable?
- (d) $T^{-1}(M) = BMB$ is given by which matrix B ?
- (e) Find the matrix for T^{-1} .

- (f) Find a matrix M so that $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} M \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}.$