Assignment 9 (Due 11/8)

§ 7.1

15) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ $T(M) = AM$

Want: I not in range of T.

If there were a matrix M such that $T(M) = I$, then $AM = I$ and A would have an inverse (namely M).

But A is singular. So no such M exists.

$M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$ has $T(M) = 0$.

$AM = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

18) $T(M) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}[M]\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Suppose $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $T(M) = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

So $M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $T(M) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$.

M is in the kernel if $M = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ (a, c, d in $\mathbb{R}$)

A matrix B is in the range of T if it is of the form $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ ($b \in \mathbb{R}$).

§ 7.2

5) $T(v_1) = w_2$, $T(v_2) = T(v_3) = w_1 + w_3$

$$A = \begin{array}{c} \begin{array}{ccc} T(v_1) & T(v_2) & T(v_3) \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \leftarrow w_1 \\ \leftarrow w_2 \\ \leftarrow w_3 \end{array}$$

$A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$T(v_1 + v_2 + v_3) = 2w_1 + w_2 + 2w_3$.

6) Since $T(v_2) = T(v_3)$, solutions to $T(v) = 0$ are $v = c(v_2 - v_3)$ (c a scalar).

The nullspace of A is spanned by $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

<u>All</u> solutions to $T(v) = w_2$ are $v = x_p + x_n = v_1 + c(v_2 - v_3)$

7) $\left[\begin{array}{ccc|c} 0 & 1 & 1 & a \\ 1 & 0 & 0 & b \\ 0 & 1 & 1 & c \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & b \\ 0 & 1 & 1 & a \\ 0 & 1 & 1 & c \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & b \\ 0 & 1 & 1 & a \\ 0 & 0 & 0 & c-a \end{array}\right]$

So $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is not in $C(A)$ if $c - a \neq 0$, i.e. if $c \neq a$.

For example, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in the column space of A.

Correspondingly, w_1 is not in the range of T.

(otherwise, a linear combination $c w_2 + d(w_1 + w_3)$ would be w_1)

Handout

1) $Q = \begin{bmatrix} \cos\theta & -s \cdot \sin\theta \\ \sin\theta & s \cdot \cos\theta \end{bmatrix}$ θ real, $s = \pm 1$.

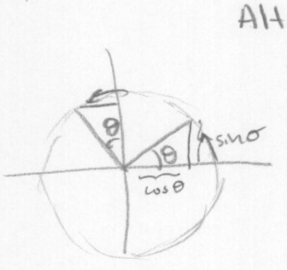
a) $\det Q = (s\cos\theta)(\cos\theta) - (\sin\theta)(-s\sin\theta) = s\cos^2\theta + s\sin^2\theta = s$.

b) According to p. 231, a rotation matrix looks like $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$,

so $\det Q = 1$. (Also, if $\det Q = 1$, then $s = 1$ and Q is a rotation).

Alternatively, choose the standard bases $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

A rotation by θ sends $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, so the matrix must be

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (with $\det = 1$)

1 continued

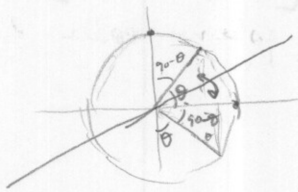
c) A (2×2) reflection matrix is a symmetric orthogonal matrix with eigenvalues 1 and -1

For Q to be symmetric, we must have $s=-1$. so $Q=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$

This Q has eigenvalues 1 and -1: characteristic polynomial $=\lambda^2-\cos^2\theta-\sin^2\theta$
$=\lambda^2-1=(\lambda-1)(\lambda+1)$

So if Q is a reflection matrix, Q must have $s=-1$, but also if $s=-1$, Q is a reflection matrix.

Alternatively, if the reflection takes $\begin{bmatrix}1\\0\end{bmatrix}$ to $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$, then it must be reflection across the line that makes a $\frac{\theta}{2}$ degree angle with the positive x-axis.



Then $\begin{bmatrix}0\\1\end{bmatrix}$ gets sent to $\begin{bmatrix}\sin\theta\\ -\cos\theta\end{bmatrix}$ and $Q=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$

d) If Q_1 and Q_2 are reflections, $Q=Q_1Q_2$ has determinant $=+1$, so it should be a rotation (since Q is still orthogonal)

We can check $Q_1=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$, $Q_2=\begin{bmatrix}\cos\varphi & \sin\varphi\\ \sin\varphi & -\cos\varphi\end{bmatrix}$

$$Q_1Q_2=\begin{bmatrix}\cos\theta\cos\varphi+\sin\theta\sin\varphi & \cos\theta\sin\varphi-\sin\theta\cos\varphi\\ \sin\theta\cos\varphi-\cos\theta\sin\varphi & \sin\theta\sin\varphi+\cos\theta\cos\varphi\end{bmatrix}$$

$$=\begin{bmatrix}\cos(\theta-\varphi) & -\sin(\theta-\varphi)\\ \sin(\theta-\varphi) & \cos(\theta-\varphi)\end{bmatrix}$$

e) Q a rotation. Show: there exist reflections Q_1, Q_2 so that $Q=Q_1Q_2$.

Let Q_1 be an arbitrary reflection. Then if $Q_1^TQ=Q_2$,

$Q_1Q_2=Q_1Q_1^TQ=Q$.

So we only need to show that Q_1^TQ is a reflection

$$Q_1^TQ=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}=\begin{bmatrix}\cos\theta\cos\varphi+\sin\theta\sin\varphi & \sin\theta\cos\varphi-\cos\theta\sin\varphi\\ \sin\theta\cos\varphi-\cos\theta\sin\varphi & -\sin\theta\sin\varphi-\cos\theta\cos\varphi\end{bmatrix}$$

$$=\begin{bmatrix}\cos(\theta-\varphi) & \sin(\theta-\varphi)\\ \sin(\theta-\varphi) & -\cos(\theta-\varphi)\end{bmatrix}$$

(or use $\det(Q_1^TQ)=-1$, so Q_1^TQ is a reflection)

So Q_1^TQ is a reflection and we can set it to be Q_2.

(Alternatively, choose θ and φ so that Q is a rotation by $\theta-\varphi$)

2) $B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

$T(C) = BC$ (2x2 matrices $\longrightarrow$ 3x2 matrices)

a)

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{bmatrix} = E_1 + 2E_3 + 3E_5$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix} = E_2 + 2E_4 + 3E_6$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} = E_1 + E_3 + E_5$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = E_2 + E_4 + E_6$$

So $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$.

b) rank (T) = rank (A)

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

rank (A) = 4, so rank (T) = 4.

c) $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ $\quad S$: 3x2 matrices $\longrightarrow \mathbb{R}^2$ by $S(H) = H^T \vec{v}$.

$$S\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So $A' = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{bmatrix}$

d) Rank A' is 2, so $\dim N(A') = 4$. Thus, the dimension of the kernel of S is 4.

e) Want: range of T = kernel of S.

First: range $T \subset$ kernel of S.

If H is in the range of T, $H = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} C$ for some 2×2 matrix C.

Then $S(H) = H^T \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = C^T \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = C^T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So if H is in the range of T, H is in the kernel of S.
But range of T and kernel of S have the same dimension.
So they must be equal.

3) $T(C) = C^T$ (C a 2×2 matrix).

a) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) $\det A = -1$ and trace $A = 2$. char. poly $= (1-\lambda) \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)^2 \lambda^2 - (1-\lambda)^2$

$= (1-\lambda)^2 (\lambda-1)(\lambda+1)$.

So $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda_4 = -1$ are the eigenvalues.

eigenvectors: $\vec{x}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\vec{x}_4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

c) T is diagonalizable. (It has a basis of eigenvectors)

4) $B = \begin{bmatrix} 5 & 2 \\ 0 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$.

$T(C) = BC + CD$.

a) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 5 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 5 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 5 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 5 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$.

So $A = \begin{bmatrix} 6 & 3 & 2 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b) Eigenvalues of T:

char poly of A: $\det \begin{bmatrix} 6-\lambda & 3 & 2 & 0 \\ 0 & 3-\lambda & 0 & 2 \\ 0 & 0 & 3-\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} = \lambda \det \begin{vmatrix} 6-\lambda & 3 & 2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$

$= \lambda(6-\lambda)(3-\lambda)^2$

Eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 6$, $\lambda_3 = \lambda_4 = 3$.

c) Is T diagonalizable?

Check: Are there 2 independent eigenvectors with eigenvalue 3?

$A - 3I = \begin{bmatrix} 3 & 3 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 3 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

free variables

$N(A-3I)$ has dimension 2, so we have a basis of eigenvectors

Yes, T is diagonalizable.

5) $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

$T(C) = BC + C^T D$

a) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} = 2E_1 + E_3$

$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$

$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}.$

So $A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

b) $\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 2.$

We know $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \text{trace}(A) = 5.$

So $\lambda_4 = 3$

c) Find C so that $BC + C^T D = 2C.$

$A - 2I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

so $C = -E_1 + E_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ has $BC + C^T D = 2C$

6) $\mathcal{P}_2$ = polynomials of degree ≤ 2.

$T(f) = g$, where $g(t) = f(t) + 5f'(t) + (1+t^2)f''(t)$.

a) $T(1) = 1 + 5 \cdot 0 + (1+t^2) \cdot 0 = 1$

$T(t) = t + 5 \cdot 1 + (1+t^2) \cdot 0 = t+5$

$T(t^2) = t^2 + 5(2t) + (1+t^2)(2) = 3t^2 + 10t + 2$.

So $A = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 10 \\ 0 & 0 & 3 \end{bmatrix}$.

b) Eigenvalues for T: $\det \begin{bmatrix} 1-\lambda & 5 & 2 \\ 0 & 1-\lambda & 10 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (1-\lambda)^2(3-\lambda)$

Eigenvalues: $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 3$.

c) Is there an f in $\mathcal{P}_2$ so that $f(t) + 5f'(t) + (1+t^2)f''(t) = 3t^2$?

This translates to is $\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ in $C(A)$?

$$\left[\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 0 & 1 & 10 & 0 \\ 0 & 0 & 3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 0 & 1 & 10 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 0 & -2 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 48 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Yes, $f(t) = t^2 - 10t + 48$ has $T(f) = 3t^2$

This f is unique, since A has full rank.