Assignment 8 (Due 11/1/12)

§6.4

3) $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

Eigenvalues: characteristic polynomial is $\det \begin{bmatrix} 2-\lambda & 2 & 2 \\ 2 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{bmatrix}$

$$= \lambda^2(2-\lambda) + 4\lambda + 4\lambda$$

$$= -\lambda^3 + 2\lambda^2 + 8\lambda = -\lambda(\lambda^2 - 2\lambda - 8)$$

$$= -\lambda(\lambda-4)(\lambda+2)$$

$\lambda_1 = 0, \quad \lambda_2 = 4, \quad \lambda_3 = -2.$

Eigenvectors: For $\lambda_1 = 0$: Solve $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \vec{x} = 0 \rightarrow \vec{x}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

For $\lambda_2 = 4$: Solve $\begin{bmatrix} -2 & 2 & 2 \\ 2 & -4 & 0 \\ 2 & 0 & -4 \end{bmatrix} \vec{x} = 0$

$$\begin{bmatrix} -2 & 2 & 2 \\ 2 & -4 & 0 \\ 2 & 0 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 2 & 2 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 2 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda_3 = -2$: Solve $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \vec{x} = 0$

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\vec{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Unit eigenvectors: $\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$

5) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$

Eigenvalues: char. poly is $\det \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{bmatrix} = (\lambda^2-1)(-\lambda) - 4(-1-\lambda) - 4(1-\lambda)$

$= -\lambda^3 + \lambda + 8\lambda = -\lambda(\lambda^2-9) = -\lambda(\lambda+3)(\lambda-3).$

Eigenvectors:

$\lambda_1 = 0$: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \vec{x} = 0$

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ So $\vec{x}_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$: $\begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{bmatrix} \vec{x} = 0$

$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{bmatrix} \to \begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 0 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ So $\vec{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$\lambda_3 = -3$: $\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \end{bmatrix} \vec{x} = 0$

$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \end{bmatrix} \to \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ So $\vec{x}_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

$Q = \frac{1}{3} \begin{bmatrix} -2 & 2 & -1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ diagonalizes A.

11) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Eigenvalues: char. poly $= \det \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 9 - 1$

$= (\lambda-2)(\lambda-4)$

Eigenvectors: $\lambda_1 = 2$: Solve $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = 0 \longrightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$

$\lambda_2 = 4$: Solve $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} = 0 \longrightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$.

So $A = 2\, \vec{x}_1 \vec{x}_1^T + 4\, x_2 x_2^T$

$= \frac{2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} [1, -1] + \frac{4}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \;\; 1]$

$= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

11 cont.

$B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ Eigenvalues: char. poly = det $\begin{bmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{bmatrix} = \lambda^2 - 25\lambda + 144 - 144$

$= \lambda(\lambda - 25)$.

Eigenvectors:

$\lambda_1 = 0$: Solve $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \vec{x} = 0$: $\vec{x}_1 = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \cdot \frac{1}{5}$

$\lambda_2 = 25$: Solve $\begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix} \vec{x} = 0$ $\vec{x}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \frac{1}{5}$

So $B = 0 \vec{x}_1 \vec{x}_1^T + 25 x_2 x_2^T$

$= 25 \cdot \frac{1}{25} \begin{bmatrix} 3 \\ 4 \end{bmatrix} [3 \ \ 4] = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$

16) $B = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$, $B\vec{x} = \lambda \vec{x}$ If $\vec{x} = \begin{bmatrix} \vec{y} \\ \vec{z} \end{bmatrix}$, $A\vec{z} = \lambda \vec{y}$, $A^T \vec{y} = \lambda \vec{z}$

a) $-\lambda$ is also an eigenvalue with eigenvector $\begin{bmatrix} y \\ -z \end{bmatrix}$ since

$B \begin{bmatrix} y \\ -z \end{bmatrix} = \begin{bmatrix} A(-z) \\ A^T y \end{bmatrix} = \begin{bmatrix} -\lambda \vec{y} \\ \lambda \vec{z} \end{bmatrix} = -\lambda \begin{bmatrix} \vec{y} \\ -\vec{z} \end{bmatrix}$ ✓

b) $A^T A \vec{z} = A^T(\lambda \vec{y}) = \lambda A^T \vec{y} = \lambda^2 \vec{z}$, so λ^2 is an eigenvalue of $A^T A$.

c) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find all 4 eigenvalues and eigenvectors of B.

eigenvectors of A: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ both with eigenvalue 1.
(same for A^T)

If λ is an eigenvalue for B, λ^2 is an eigenvalue for $A^T A = I$,
so the only eigenvalues B can have are ± 1.

But trace of B = 0, so there must be as many +1's as −1's.

So $\boxed{\lambda_1 = \lambda_2 = 1 \text{ and } \lambda_3 = \lambda_4 = -1}$

Eigenvectors: Solve $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \vec{x} = 0$

$\boxed{\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$

Solve $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \vec{x} = 0$

$\boxed{\vec{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$

16 d) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$C = A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which has eigenvalue $\lambda_1 = \lambda_2 = 1$.

$B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$ has eigenvalues $\lambda_1 = \lambda_2 = 1$
$\lambda_3 = \lambda_4 = -1$
$\lambda_5 = 0$.

$\lambda_5^2 = 0^2 = 0$ is not an eigenvalue for C

The problem is that $\vec{x}_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, so $\vec{z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in this case.

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not an eigenvector for C since we always take eigenvectors to be nonzero vectors.

23) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is invertible, orthogonal, permutation, diagonalizable and Markov.

A has QR, $S\Lambda S^{-1}$, $Q\Lambda Q^T$ factorizations

$B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is projection, diagonalizable and Markov.

(diagonalizable because eigenvalues are $\lambda_1 = \lambda_2 = 0$ (rank = 1) and $\lambda_3 = 1$

with eigenvectors $\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$)

B has LU, $S\Lambda S^{-1}$ and $Q\Lambda Q^T$ factorizations.

24) $\begin{bmatrix} 2 & b \\ 1 & 0 \end{bmatrix} = A$.

$b = 1$ makes $Q\Lambda Q^T = A$ possible

For $S\Lambda S^{-1}$ to be impossible, we need only one eigenvector i.e. a repeated eigenvalue, and only one eigenvector

Eigenvalues are solutions to $\lambda^2 - 2\lambda - b = 0$, $\frac{2 \pm \sqrt{4+4b}}{2} = 1 \pm \sqrt{1+b}$

So if $b = -1$, we get a repeated eigenvalue

$b = -1$ makes $S\Lambda S^{-1}$ impossible.

$b = 0$ makes A^{-1} impossible.

§6.5

9) $[x_1\ x_2\ x_3]\, A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2 = [X\ Y\ Z] \begin{bmatrix} 24 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$

$\left(\text{if } X = \dfrac{x_1 - x_2 + 2x_3}{\sqrt{6}}\right)$ (normalized)

Eigenvalues of A: 24, 0, 0

Eigen vectors: $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, vectors perpendicular to $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, like $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

So an orthonormal basis of eigenvectors is $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

and $A = Q \Lambda Q^T = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & \sqrt{3} & \sqrt{2} \\ -1 & \sqrt{3} & -\sqrt{2} \\ 2 & 0 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 24 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & -1 & 2 \\ \sqrt{3} & \sqrt{3} & 0 \\ \sqrt{2} & -\sqrt{2} & -\sqrt{2} \end{bmatrix}$

$= \frac{1}{6} \begin{bmatrix} 24 & 0 & 0 \\ -24 & 0 & 0 \\ 48 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ \sqrt{3} & \sqrt{3} & 0 \\ \sqrt{2} & -\sqrt{2} & -\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$

So eigenvalues of A are 24, 0, 0

with eigen vectors $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

Rank of A = 1

pivots of A: 4

determinant: 0

10) $\vec{x}^T A \vec{x} = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. A is positive definite because $\vec{x}^T A \vec{x} = (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_1^2 + x_3^2$ which is greater than 0 if $(x_1, x_2, x_3) \neq (0,0,0)$.

$\vec{x}^T B \vec{x} = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3)$

$B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$. $\vec{x}^T B \vec{x} = (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2 \geq 0$

This is positive semidefinite since if $x_1 = x_2 = x_3$, $\vec{x}^T B \vec{x} = 0$, but otherwise $\vec{x}^T B \vec{x} > 0$.

11) $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}$

$\det [2] = 2$

$\det \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} = 10 - 4 = 6$

$\det \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix} = 80 - 18 - 32 = 30$

So A is positive definite

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

2^{ND} pivot $= 3 = \frac{6}{2}$

3^{RD} pivot $= 5 = \frac{30}{6}$ ✓

12) $A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$ determinants are c, $c^2 - 1$ and $c^3 + 1 + 1 - c - c - c$

A is positive definite if $c > 0$, $c^2 - 1 > 0$ and $c^3 - 3c + 2 > 0$.

$c^2 - 1 > 0$ if $(c-1)(c+1) > 0$

So $c > 1$ or $c < -1$

$c^3 - 3c + 2 = (c+2)(c^2 - 2c + 1) = (c+2)(c-1)^2$

So $c^3 - 3c + 2 > 0$ if

So A is positive definite if $c > 1$

12 cont.

$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}$ determinants are 1, $d-4$, $5d+24+24-9d-16-20$

So B is positive definite if $d-4>0$ and $-4d+12>0$

So $d>4$ and $d<3 \Rightarrow$ B is never positive definite

21.) $A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix}$ Eigenvalues are roots of

$$\det \begin{bmatrix} s-\lambda & -4 & -4 \\ -4 & s-\lambda & -4 \\ -4 & -4 & s-\lambda \end{bmatrix} = (s-\lambda)^3 - 64 - 64 - 16(s-\lambda) - 16(s-\lambda) - 16(s-\lambda)$$

$$= (s-\lambda)^3 - 48(s-\lambda) - 128$$

$$= \left[(s-\lambda)-8\right]\left[(s-\lambda)+4\right]^2$$

(since $x^3 - 48x - 128 = (x-8)(x+4)^2$)

So eigenvalues are

$\lambda = s-8$

$\lambda = s+4$

For A to be positive definite, need $4+s>0$ and $s-8>0$

So $\boxed{s>-4 \text{ and } s>8}$ (so $s>8$)

$B = \begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix}$ Eigenvalues are roots of

$$\det \begin{bmatrix} t-\lambda & 3 & 0 \\ 3 & t-\lambda & 4 \\ 0 & 4 & t-\lambda \end{bmatrix} = (t-\lambda)^3 - 16(t-\lambda) - 9(t-\lambda)$$

$$= (t-\lambda)\left((t-\lambda)^2 - 25\right) = (t-\lambda)(t-\lambda-5)(t-\lambda+5)$$

Eigenvalues are t, $t-5$, $t+5$.

For B to be positive definite, need $t>0$, $t-5>0$ and $t+5>0$,

ie $t>0$, $t>5$ and $t>-5$.

so $\boxed{t>5}$

22) $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

Eigenvalues: $\det \begin{bmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{bmatrix} = 25 - 10\lambda + \lambda^2 - 16 = \lambda^2 - 10\lambda + 9$
$= (\lambda-1)(\lambda-9)$.

Eigenvectors: $\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \vec{x} = 0 \longrightarrow \vec{x}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \vec{x} = 0 \longrightarrow x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

$R = Q \Lambda^{1/2} Q^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ ✓

$A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$.

Eigenvalues: $\det \begin{bmatrix} 10-\lambda & 6 \\ 6 & 10-\lambda \end{bmatrix} = 100 - 20\lambda + \lambda^2 - 36 = \lambda^2 - 20\lambda + 64$
$= (\lambda - 16)(\lambda - 4)$

Eigenvectors: $\begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix} \vec{x} = 0 \longrightarrow \vec{x}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \vec{x} = 0 \rightsquigarrow \vec{x}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

So $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 16 & 0 \\ 0 & 4 \end{bmatrix}$.

$R = Q \Lambda^{1/2} Q^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$ ✓

24) $x^2 + xy + y^2 = 1$.

$$(x\,y)\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = 1.$$

Eigenvalues: $\det\begin{bmatrix}1-\lambda & \frac{1}{2}\\ \frac{1}{2} & 1-\lambda\end{bmatrix} = 1 - 2\lambda + \lambda^2 - \frac{1}{4} = \lambda^2 - 2\lambda + \frac{3}{4}$

$= (\lambda - \frac{3}{2})(\lambda - \frac{1}{2})$

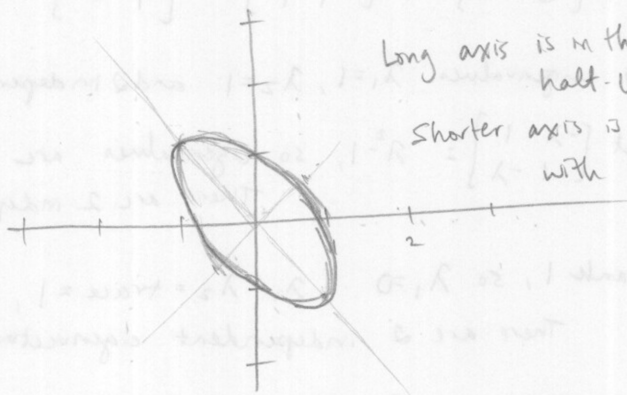
Eigenvectors: $\begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}\vec{x} = 0 \rightsquigarrow \vec{x}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$

$\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\vec{x} = 0 \rightsquigarrow \vec{x}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$

So $\vec{x}^T A \vec{x} = x^T \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{3}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix}x+y, x-y\end{bmatrix}\begin{bmatrix}\frac{3}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}x+y\\ x-y\end{bmatrix}$

$= \frac{3}{2}\left(\frac{x+y}{\sqrt{2}}\right)^2 + \frac{1}{2}\left(\frac{x-y}{\sqrt{2}}\right)^2 = 1$



Long axis is in the $(-1,1)$ direction with half-length $\sqrt{2}$

Shorter axis is in the $(1,1)$ direction with half-length $\frac{\sqrt{2}}{\sqrt{3}}$

26) $A = \begin{bmatrix}9 & 0 & 0\\ 0 & 1 & 2\\ 0 & 2 & 8\end{bmatrix}$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -2 & 1\end{bmatrix}\begin{bmatrix}9 & 0 & 0\\ 0 & 1 & 2\\ 0 & 2 & 8\end{bmatrix} = \begin{bmatrix}9 & 0 & 0\\ 0 & 1 & 2\\ 0 & 0 & 4\end{bmatrix} = \begin{bmatrix}9 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}$$

So $A = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2 & 1\end{bmatrix}\begin{bmatrix}9 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}$

$C^T = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2 & 1\end{bmatrix}\begin{bmatrix}3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix} = \begin{bmatrix}3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2 & 2\end{bmatrix}$

$C = \begin{bmatrix}3 & 0 & 0\\ 0 & 1 & 2\\ 0 & 0 & 2\end{bmatrix}$

26 (cont) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

So $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

and $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$.

§ 6.6

5) Which are similar?

$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ $A_5 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $A_6 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$A_1 = I$ and it has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1$ and 2 independent eigenvectors.

A_2: Eigenvalues: $\det\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1$, so eigenvalues are ± 1.
There are 2 independent eigenvectors.

A_3: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has rank 1, so $\lambda_1 = 0$ $\lambda_1 + \lambda_2 = \text{trace} = 1$, so $\lambda_2 = 1$
There are 2 independent eigenvectors.

$A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ has rank 1, so $\lambda_1 = 0$. Again $\lambda_1 + \lambda_2 = \text{trace} = 1$, so $\lambda_2 = 1$
There are 2 independent eigenvectors.

$A_5 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ has rank 1, so $\lambda_1 = 0$ and $\lambda_2 = 1$ (since $\lambda_1 + \lambda_2 = 1$)
There are 2 independent eigenvectors.

$A_6 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ has rank 1, so $\lambda_1 = 0$ and $\lambda_2 = 1$ (as above).
There are 2 independent eigenvectors.

So A_3, A_4, A_5, A_6 are similar.

A_1 is not similar to any of these other matrices.

A_2 is not similar to any of the other matrices either.