Assignment 5 (Due 10/11)

§4.1

3) a) Column space contains $\begin{bmatrix}1\\2\\-3\end{bmatrix}$, $\begin{bmatrix}2\\-3\\5\end{bmatrix}$ and nullspace contains $\begin{bmatrix}1\\1\\1\end{bmatrix}$

Put $\begin{bmatrix}1\\2\\-3\end{bmatrix}$ and $\begin{bmatrix}2\\-3\\5\end{bmatrix}$ in as columns to force the column space requirement.

$$A = \begin{bmatrix}1 & 2 & a\\ 2 & -3 & b\\ -3 & 5 & c\end{bmatrix}.$$

For the nullspace requirement, need $\left.\begin{array}{l}1+2+a=0\\ 2-3+b=0\\ -3+5+c=0\end{array}\right\} \to \begin{array}{l}a=-3\\ b=1\\ c=-2.\end{array}$

So $A = \begin{bmatrix}1 & 2 & -3\\ 2 & -3 & 1\\ -3 & 5 & -2\end{bmatrix}$ works.

b) Row space contains $\begin{bmatrix}1\\2\\-3\end{bmatrix}$ and $\begin{bmatrix}2\\-3\\5\end{bmatrix}$, nullspace contains $\begin{bmatrix}1\\1\\1\end{bmatrix}$

There is no matrix satisfying these conditions.

$C(A^T) \perp N(A)$, but $\begin{bmatrix}2\\-3\\5\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$ aren't perpendicular

c) $A\vec{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ has a solution and $A^T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

$A^T\begin{bmatrix}1\\0\\0\end{bmatrix}$ means A^T has 1^{ST} column $= \begin{bmatrix}0\\0\\0\end{bmatrix}$

and A has first row $(0 \;\; 0 \;\; 0)$.

But if $A = \begin{bmatrix}0 & 0 & 0\\ a & b & c\\ d & e & f\end{bmatrix}$, $A\vec{x}$ will always have first component 0 (i.e. will always be of the form $\begin{bmatrix}0\\x\\y\end{bmatrix}$)

So $A\vec{x}$ can't be $\begin{bmatrix}1\\1\\1\end{bmatrix}$

Therefore, there is no matrix satisfying these conditions.

d) Every row is orthogonal to every column (and A not the zero matrix)

This would mean that $A^2=0$. So A could be

$$A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

3) e) Columns add up to a column of zeros, rows add to a row of 1's.

This would mean that $A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, so $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is in $N(A)$.

and $(1, 1, 1)$ is in the row space $(C(A^T))$.

But $N(A) \perp C(A^T)$, and $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is not perpendicular to itself.

So no such matrix exists.

6)

$$x + 2y + 2z = 5$$
$$2x + 2y + 3z = 5$$
$$3x + 4y + 5z = 9$$

If $y_1 = 1$, $y_2 = 1$ and $y_3 = -1$, then

y_1 (1ST eqn) + y_2 (2ND eqn) + y_3 (3RD eqn) gives $0 = 1$.

This means $(1, 1, -1)$ is in the left nullspace of the matrix of coefficients $A = \begin{bmatrix}1 & 2 & 2\\ 2 & 2 & 3\\ 3 & 4 & 5\end{bmatrix}$.

$y^T\vec{b} = 1$, so we found a vector in the left nullspace not perpendicular to the column space.

Since this is impossible, there is no solution.

22) P the plane of vectors in $\mathbb{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$.

A basis for $P^\perp$: $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$.

$A = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$ is a matrix with P as the nullspace.

§ 4.2

11) a) $A = \begin{bmatrix}1 & 1\\ 0 & 1\\ 0 & 0\end{bmatrix}$, $\vec{b} = \begin{bmatrix}2\\3\\4\end{bmatrix}$.

$$A^T\vec{b} = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix}$$

$$A^TA = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\\ 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$$

Want $\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix}$: $\left[\begin{array}{cc|c}1 & 1 & 2\\ 1 & 2 & 5\end{array}\right] \to \left[\begin{array}{cc|c}1 & 1 & 2\\ 0 & 1 & 3\end{array}\right] \to \left[\begin{array}{cc|c}1 & 0 & -1\\ 0 & 1 & 3\end{array}\right]$

So $p = A\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix} = \boxed{\begin{bmatrix}2\\3\\0\end{bmatrix}}$

13) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad P = \underset{4\times3}{A}\,(\underset{3\times4}{A^T}\,\underset{4\times3}{A})^{-1}\,\underset{3\times4}{A^T}$, so P is 4 by 4

projection of $\vec{b} = (1, 2, 3, 4)$ is $(1, 2, 3, 0)$.

P is 4 by 4. $\quad A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so $P = AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

16) What linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ is closest to $\vec{b} = (2, 1, 1)$.

In other words, project $\vec{b}$ onto the space spanned by $(1, 2, -1)$ and $(1, 0, 1)$;

so $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$.

$$A^TA = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$A^TA\vec{x} = A^T\vec{b}$, so $\begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

$x = \frac{1}{2}, \quad y = \frac{3}{2}$

$$\vec{p} = A\vec{x} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1) = (2, 1, 1) = \vec{b} = P\vec{b}$$

(So $\vec{b}$ was in the plane to begin with).

18) b) If P is the 3×3 projection onto the line through $(1,1,1)$, then $I-P$ is the projection matrix onto the orthogonal complement of the line through $(1,1,1)$ (i.e. the plane $x+y+z=0$).

19) Projection onto: $x-y-2z=0$

Two vectors in this plane: $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}$ $\quad A=\begin{bmatrix}1&2\\1&0\\0&1\end{bmatrix}$

$$P=A(A^TA)^{-1}A^T$$

$$A^TA=\begin{bmatrix}1&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&2\\1&0\\0&1\end{bmatrix}=\begin{bmatrix}2&2\\2&5\end{bmatrix}$$

$$(A^TA)^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}=\frac{1}{6}\begin{bmatrix}5&-2\\-2&2\end{bmatrix}=\begin{bmatrix}5/6&-1/3\\-1/3&1/3\end{bmatrix}.$$

$$\text{So } P=\begin{bmatrix}1&2\\1&0\\0&1\end{bmatrix}\begin{bmatrix}5/6&-1/3\\-1/3&1/3\end{bmatrix}\begin{bmatrix}1&1&0\\2&0&1\end{bmatrix}$$

$$=\begin{bmatrix}1/6&1/3\\5/6&-1/3\\-1/3&1/3\end{bmatrix}\begin{bmatrix}1&1&0\\2&0&1\end{bmatrix}=\begin{bmatrix}5/6&1/6&1/3\\1/6&5/6&-1/3\\1/3&-1/3&1/3\end{bmatrix}$$

§4.3

1) $b=0,\ 8,\ 8,\ 20$; $t=0,\ 1,\ 3,\ 4$

$C+D\cdot0=0$

$C+D\cdot1=8$

$C+D\cdot3=8$

$C+D\cdot4=20$

$$\rightsquigarrow \underbrace{\begin{bmatrix}1&0\\1&1\\1&3\\1&4\end{bmatrix}}_{A}\begin{bmatrix}C\\D\end{bmatrix}=\underbrace{\begin{bmatrix}0\\8\\8\\20\end{bmatrix}}_{\vec{b}}$$

$$A^TA=\begin{bmatrix}1&1&1&1\\0&1&3&4\end{bmatrix}\begin{bmatrix}1&0\\1&1\\1&3\\1&4\end{bmatrix}=\begin{bmatrix}4&8\\8&26\end{bmatrix}$$

$$A^T\vec{b}=\begin{bmatrix}1&1&1&1\\0&1&3&4\end{bmatrix}\begin{bmatrix}0\\8\\8\\20\end{bmatrix}=\begin{bmatrix}36\\112\end{bmatrix}$$

$$A^TA\vec{x}=A^T\vec{b}:\ \left[\begin{array}{cc|c}4&8&36\\8&26&112\end{array}\right]\to\left[\begin{array}{cc|c}4&8&36\\0&10&40\end{array}\right]\to\left[\begin{array}{cc|c}1&2&9\\0&1&4\end{array}\right]\to\left[\begin{array}{cc|c}1&0&1\\0&1&4\end{array}\right]$$

So $\vec{p}=A\vec{x}=\begin{bmatrix}1&0\\1&1\\1&3\\1&4\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix}=\begin{bmatrix}1\\5\\13\\17\end{bmatrix}$ and

$p_1=1 \quad e_1=-1$

$p_2=5 \quad e_2=3$

$p_3=13 \quad e_3=-5$

$p_4=17 \quad e_4=3$

$E=1+9+25+9=\boxed{44}$ is the minimum.

3) $\vec{e} = (-1, 3, -5, 3)$

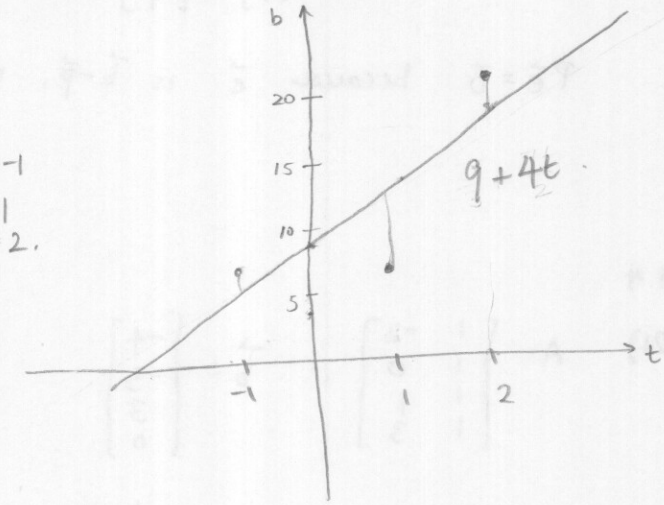
$\vec{e} \cdot (1, 1, 1, 1) = -1 + 3 - 5 + 3 = 0$

$\vec{e} \cdot (0, 1, 3, 4) = 0 + 3 - 15 + 12 = 0$

So $\vec{e}$ is orthogonal to the columns of A.

The shortest distance $\|\vec{e}\|$ from $\vec{b}$ to the column space of A

is $\sqrt{44} = \boxed{2\sqrt{11}}$



17) $b = C + Dt$ goes through $b = 7$ at $t = -1$

$b = 7$ at $t = 1$

$b = 21$ at $t = 2$.

$C + D(-1) = 7$

$C + D = 7$

$C + 2D = 21$

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$

Want $A \begin{bmatrix} C \\ D \end{bmatrix} = \vec{b}$, but this is impossible, so solve $A^TA \begin{bmatrix} C \\ D \end{bmatrix} = A^T\vec{b}$ instead.

$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

$\left[\begin{array}{cc|c} 3 & 2 & 35 \\ 2 & 6 & 42 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 3 & 2 & 35 \\ 0 & 14/3 & 56/3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 2 & 35 \\ 0 & 1 & 4 \end{array}\right]$

$\longrightarrow \left[\begin{array}{cc|c} 3 & 0 & 27 \\ 0 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 4 \end{array}\right]$

$\hat{x} = (C, D) = (9, 4)$

18) $\vec{p} = A\vec{x}$ from #17:

$$\vec{p} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix}$$

$$\vec{e} = \vec{b} - \vec{p} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} - \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}$$

$P\vec{e} = \vec{0}$ because $\vec{e}$ is $\vec{b}-\vec{p}$, so $P\vec{e} = P(\vec{b}-\vec{p}) = P\vec{b} - P^2\vec{b}$
$= P\vec{b} - P\vec{b} = \vec{0}$. ($P^2 = P$ if P is a proj matrix).

§ 4.4

21) $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix}$.

We want an orthonormal basis for the column space of A.

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

Replace $\vec{a}_2$ by $\vec{a}_2 - \vec{p}$ where $\vec{p}$ is the projection of $\vec{a}_2$ onto the line spanned by $\vec{a}_1$.

$$\vec{p} = \frac{\vec{a}_1\vec{a}_1^T}{\vec{a}_1^T\vec{a}_1}\vec{a}_2 = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} =$$

So replace $\vec{a}_2$ by $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ -1/2 \\ 1/2 \\ 5/2 \end{bmatrix} = \vec{a}_2'$

Normalize: replace $\vec{a}_1$ by $\frac{\vec{a}_1}{\|\vec{a}_1\|}$ and $\vec{a}_2'$ by $\frac{\vec{a}_2'}{\|\vec{a}_2'\|}$

$$\vec{q}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{q}_2 = \frac{1}{\sqrt{13}}\begin{bmatrix} -5/2 \\ -1/2 \\ 1/2 \\ 5/2 \end{bmatrix}$$

The projection of $\vec{b}$ onto this space is $Q\underbrace{(Q^TQ)^{-1}}_{I}Q^T\vec{b} = QQ^T\vec{b}$.

21 continued

$$QQ^T = \begin{bmatrix} \frac{1}{2} & -\frac{5}{2\sqrt{13}} \\ \frac{1}{2} & -\frac{1}{2\sqrt{13}} \\ \frac{1}{2} & \frac{1}{2\sqrt{13}} \\ \frac{1}{2} & \frac{5}{2\sqrt{13}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{5}{2\sqrt{13}} & -\frac{1}{2\sqrt{13}} & \frac{1}{2\sqrt{13}} & \frac{5}{2\sqrt{13}} \end{bmatrix} = \frac{1}{(2\sqrt{13})^2} \begin{bmatrix} \sqrt{13} & -5 \\ \sqrt{13} & -1 \\ \sqrt{13} & 1 \\ \sqrt{13} & 5 \end{bmatrix} \begin{bmatrix} \sqrt{13} & \sqrt{13} & \sqrt{13} & \sqrt{13} \\ -5 & -1 & 1 & 5 \end{bmatrix}$$

$$= \frac{1}{52} \cdot \begin{bmatrix} 38 & 18 & 8 & -12 \\ 18 & 14 & 12 & 8 \\ 8 & 12 & 14 & 18 \\ -12 & 8 & 18 & 38 \end{bmatrix}$$

$$QQ^T\vec{b} = \frac{1}{26} \begin{bmatrix} 19 & 9 & 4 & -6 \\ 9 & 7 & 6 & 4 \\ 4 & 6 & 7 & 9 \\ -6 & 4 & 9 & 19 \end{bmatrix} \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -7/2 \\ -3/2 \\ -1/2 \\ 3/2 \end{bmatrix} = \vec{p}}$$

23) Find $\vec{q}_1, \vec{q}_2, \vec{q}_3$ as combinations of $\vec{a}, \vec{b}, \vec{c}$ and write A as QR.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

Replace $\vec{b}$ by $\vec{B} = \vec{b} - \vec{p} = \vec{b} - \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\vec{a} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \frac{2}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$.

Replace $\vec{c}$ by $\vec{C} = \vec{c} - \vec{p}' = \vec{c} - \frac{\vec{a}^T\vec{c}}{\vec{a}^T\vec{a}}\vec{a} - \frac{\vec{B}^T\vec{c}}{\vec{B}^T\vec{B}}\vec{B}$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{4}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{18}{9}\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

Normalize: $\vec{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{q}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{q}_1 = \vec{a}$, $\quad \vec{q}_2 = \frac{\vec{b} - 2\vec{a}}{3} = \frac{1}{3}\vec{b} - \frac{2}{3}\vec{a}$, $\quad \vec{q}_3 = \frac{\vec{c} - 4\vec{a} - 2(\vec{b} - 2\vec{a})}{5} = \frac{1}{5}\vec{c} + 0\vec{a} - \frac{2}{5}\vec{b}$

23 cont.

So $\left[\vec{a}\ \vec{b}\ \vec{c}\right] \underbrace{\begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 1/3 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}}_{R^{-1}} = \underbrace{\left[q_1\ q_2\ q_3\right]}_{Q}$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}}_{R}.$$

31. $Q = c \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$

If $c = \frac{1}{2}$, this is an orthogonal matrix.

Project $\vec{b}$ onto the 1^{ST} column: $\vec{b} = (1, 1, 1, 1)$

$$\vec{p} = \frac{\vec{a}_1^T \vec{b}}{\vec{a}_1^T \vec{a}_1} \vec{a}_1 = \frac{-1}{1} \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Project $\vec{b}$ onto the plane of the first 2 columns

Projection of $\vec{b}$ onto 2^{ND} column:

$$\frac{\vec{a}_2^T \vec{b}}{\vec{a}_2^T \vec{a}_2} \vec{a}_2 = \frac{-1}{1} \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Projection of $\vec{b}$ onto plane of 1^{ST} 2 columns:

$$\begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$