Assignment 4 (Due 10/4).

§3.4 1)

$$\left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & -1 & -1 & -2 & b_3-b_1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & 0 & 0 & 0 & b_2+b_3-2b_1 \end{array}\right]$$

- $A\vec{x}=\vec{b}$ has a solution when $b_2+b_3-2b_1=0$.
- The column space of A is a plane (the plane spanned by $\begin{bmatrix}2\\2\\2\end{bmatrix}$ and $\begin{bmatrix}4\\5\\3\end{bmatrix}$).

  Alternately, it is the plane $b_2+b_3-2b_1=0$
- The nullspace of A is the nullspace of U.

  It is spanned by special solutions s_1 and s_2.

$$s_1 = \begin{bmatrix}a\\b\\1\\0\end{bmatrix}, \quad s_2 = \begin{bmatrix}c\\d\\0\\1\end{bmatrix}.$$

$$2a+4b+6=0 \qquad 2c+4d+4=0$$
$$b+1=0 \qquad d+2=0$$
$$b=-1,\ a=-1 \qquad d=-2,\ c=2.$$

The nullspace is make up of vectors

$$m\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix} + n\begin{bmatrix}2\\-2\\0\\1\end{bmatrix}.$$

- A particular solution to $A\vec{x}=\vec{b}$ (where $\vec{b}=\begin{bmatrix}4\\3\\5\end{bmatrix}$) comes from

$$\left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We set free variables to 0. and solve

$$2x+4y=4 \qquad x=4.$$
$$y=-1$$

$$\vec{x}_p = \begin{bmatrix}4\\-1\\0\\0\end{bmatrix}$$

The complete solution is $\vec{x} = \begin{bmatrix}4\\-1\\0\\0\end{bmatrix} + m\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix} + n\begin{bmatrix}2\\-2\\0\\1\end{bmatrix}$.

1) cont.

$$[U \mid c] = \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [R \mid d].$$

6)

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 3 & b_4 - 3b_1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_4 - 3b_1 - 3(b_3 - 2b_1) \end{array}\right]$$

So this is solvable when $b_2 - 2b_1 = 0$ and $b_4 - 3b_3 + 3b_1 = 0$.

In this case, $x_2 = b_3 - 2b_1$

$x_1 + 2x_2 = b_1 \longrightarrow x_1 = b_1 + 4b_1 - 2b_3$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix}.$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & b_1 - 2b_3 + 4b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 0 & 0 & b_4 - 3b_1 - 3b_3 + 6b_1 \end{array}\right]$$

This is solvable when $b_2 - 2b_1 = 0$ and $b_4 - 3b_3 + 3b_1 = 0$.

Nullspace has special solution $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, particular solution is $\begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix}$.

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + C \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$

8) For a vector to be in the column space of A, we need to be able to solve $A\vec{x}=\vec{b}$.

a) $A=\begin{bmatrix}1&2&1\\2&6&3\\0&2&5\end{bmatrix}$

$$\left[\begin{array}{ccc|c}1&2&1&b_1\\2&6&3&b_2\\0&2&5&b_3\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&2&1&b_1\\0&2&2&b_2-2b_1\\0&2&5&b_3\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&0&-1&3b_1-b_2\\0&2&2&b_2-2b_1\\0&0&3&b_3-b_2+2b_1\end{array}\right].$$

A has full row and full column rank and is invertible.

Thus, every vector in $\mathbb{R}^3$ is in the column space.

No combination of the rows of A is $\vec{0}$.

b) $A=\begin{bmatrix}1&1&1\\1&2&4\\2&4&8\end{bmatrix}$.

$$\left[\begin{array}{ccc|c}1&1&1&b_1\\1&2&4&b_2\\2&4&8&b_3\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&1&1&b_1\\0&1&3&b_2-b_1\\0&2&6&b_3-2b_1\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&1&1&b_1\\0&1&3&b_2-b_1\\0&0&0&b_3-2b_2\end{array}\right].$$

The column space of A consists of vectors $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ with $b_3=2b_2$.

(Alternately, it is spanned by $\begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\2\\4\end{bmatrix}$)

The combinations $0\cdot(1,1,1)+-2c\,(1,2,4)+c\,(2,4,8)$ give the zero vector.

10) Construct $A\vec{x}=\vec{b}$ with $\vec{x}_p=(2,4,0)$, $x_n=$ multiples of $(1,1,1)$.

Free variable $= x_3$

$$\begin{bmatrix}1&0&a\\0&1&b\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\vec{0}\longrightarrow\begin{array}{l}a=-1\\b=-1.\end{array}$$

So $A=\begin{bmatrix}1&0&-1\\0&1&-1\end{bmatrix}$ works

Now we need $\vec{b}$, but $\vec{x}_p$ is a solution, so $\vec{b}=A\vec{x}_p$.

$$\begin{bmatrix}1&0&-1\\0&1&-1\end{bmatrix}\begin{pmatrix}2\\4\\0\end{pmatrix}=\begin{bmatrix}2\\4\end{bmatrix}.$$

So $A\vec{x}=\vec{b}$ is $\begin{bmatrix}1&0&-1\\0&1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$.

17) The largest possible rank of a 6 by 4 matrix is $\underline{4}$.

Then there is a pivot in every $\underline{\text{column}}$ of U and R.

The solution to $A\vec{x} = \vec{b}$ is $\underline{\text{unique}}$.

The nullspace of A is $\underline{\text{zero}}$.

An example is $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

24) a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ will give $A\vec{x} = \vec{b}$ with 0 or 1 solution, depending on $\vec{b}$.

b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ will give $A\vec{x} = \vec{b}$ with infinitely many solutions.

c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ will give $A\vec{x} = \vec{b}$ with 0 or ∞ solutions depending on $\vec{b}$.

d) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ will give $A\vec{x} = \vec{b}$ with 1 solution, regardless of $\vec{b}$.

§3.5

3) $U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

$\underline{\text{If } a=0}$: $2 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} b \\ d \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} c \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so columns of U are dependent.

$\underline{\text{If } d=0}$: $b \cdot \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} - a \cdot \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} c \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so columns of U are dependent.

$\underline{\text{If } f=0}$: $\left(\frac{eb-cd}{a}\right) \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + e \cdot \begin{bmatrix} b \\ d \\ 0 \end{bmatrix} - d \begin{bmatrix} c \\ e \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so columns of U are dependent.

10) $x + 2y - 3z - t = 0$ plane in $\mathbb{R}^4$.

Three independent vectors:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

There can't be four independent vectors in the plane because if there were, the plane would be all of $\mathbb{R}^4$. The plane has dimension 3.

$A = \begin{bmatrix} 1 & 2 & -3 & -1 \end{bmatrix}$ has ~~this~~ plane as its nullspace.

13) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix}$ $\quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

a) $\dim C(A) = 2$ (= rank).

b) $\dim C(U) = 2$ (= rank)

c) $\dim C(A^T) = 2$ (= rank)

d) $\dim C(U^T) = 2$ (= rank)

The row space of A and the row space of U are the same space.

22) S a 5-dim. subspace of $\mathbb{R}^6$.

a) Every basis of S can be extended to a basis for $\mathbb{R}^6$ by adding one vector. TRUE.

(Add any vector not in S).

b) Every basis for $\mathbb{R}^6$ can be reduced to a basis for S by deleting a vector FALSE.

For example, $S = \left\{ \begin{bmatrix} 0 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \right\}$

Then $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^6$.

But none are in S, so deleting one won't give a basis for S.

35) A basis for polynomials with degree ≤ 3: $\boxed{1,\ x,\ x^2,\ x^3.}$

(Any poly. with degree ≤ 3 is a linear combo of these

$$ax^3 + bx^2 + cx + d \cdot 1$$

and they are linearly independent since scalars are <u>numbers</u>, not x's).

A basis for the subspace with $p(1)=0$:

If $p(1)=0$, then that means $a+b+c+d=0$ (for $p(x)=ax^3+bx^2+cx+d$).

Basis: $\boxed{x-1,\ x^2-x,\ x^3-x^2.}$

These are linearly independent:

If $a(x-1) + b(x^2-x) + c(x^3-x^2) = 0$, then

$cx^3 + (b-c)x^2 + (a-b)x - a = 0$, so

all coefficients must be 0.

$$\left.\begin{array}{l} c=0 \\ b-c=0 \\ a-b=0 \\ a=0 \end{array}\right\} \Rightarrow b=0.$$

Thus, the only linear combo of these vectors that gives 0 is the combo with all coefficients equal to 0. This means the vectors are <u>independent</u>.

These vectors actually span the space:

Say $p(x) = ax^3 + bx^2 + cx + d$ is in the space

(so $a+b+c+d=0$).

Then $a(x^3-x^2) + (a+b)(x^2-x) + (a+b+c)(x-1)$

$= ax^3 + bx^2 + cx + -(a+b+c)$.

Since $a+b+c+d=0$, $-(a+b+c) = d$, so in fact

$a(x^3-x^2) + (a+b)(x^2-x) + (a+b+c)(x-1) = ax^3+bx^2+cx+d$
$= p(x)$.

So the vectors span, and they do form a basis.

§ 3.6

2) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

Column space: dimension = 1.

basis: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Row space: dimension = 1

basis: $(1, 2, 4)$.

Nullspace: dimension = 2.

basis: special solutions.

$s_1 = \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} b \\ 0 \\ 1 \end{bmatrix}.$

$a + 2 = 0 \qquad b + 4 = 0$

$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad s_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}.$

Left nullspace: dimension = 1.

basis: $(-2, 1)$.

$B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}$

Column space: dimension = 2.

basis: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

Row space: dimension = 2

basis: $(1, 2, 4), \ (2, 5, 8)$.

Nullspace: dimension = 1

basis: $\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$.

Left nullspace: dimension = 0

basis: .

empty set.

14) $A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

Column space: basis is 1^{ST}, 2^{ND}, 3^{RD} columns of A, which are

$\begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 13 \\ 26 \end{bmatrix}, \begin{bmatrix} 3 \\ 20 \\ 44 \end{bmatrix}$.

(Alternate, better solution: $C(A)$ is dim. 3 in $\mathbb{R}^3$, so it is all of $\mathbb{R}^3$. Basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$)

Row space: basis is $(1, 2, 3, 4)$
$(0, 1, 2, 3)$
$(0, 0, 1, 2)$.

Nullspace: basis is special solution $\begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix}$.

$a + 2b + 3c + 4 = 0 \rightarrow a = 0$
$b + 2c + 3 = 0 \rightarrow b = 1$
$c + 2 = 0 \rightarrow c = -2$

basis: $\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$

Left nullspace: basis: empty, since dim $N(A^T) = 0$.

19) a) $\begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & -2 & b_3 - 3b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & 0 & b_3 - b_2 \end{bmatrix}$

Basis for left nullspace: $(0, -1, 1)$.

b) $\begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - 2b_1 \\ 0 & 1 & b_4 - 2b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - 2b_1 \\ 0 & 0 & b_4 + b_2 - 4b_1 \end{bmatrix}$.

Basis for left nullspace:

$(-2, 0, 1, 0)$, $(-4, 1, 0, 1)$.

24) $A^T y = d$ is solvable when d is in $C(A^T)$ = row space of A.

The solution is unique when the left nullspace of $A = N(A^T)$ contains only the zero vector.

25) a) A and A^T have the same number of pivots.

TRUE. # pivots of A = rank of A = $\dim C(A) = \dim C(A^T)$.

pivots of A^T = rank of A^T = $\dim C(A^T) = \dim C((A^T)^T) = \dim C(A)$.

So A and A^T have the same number of pivots.

b) A and A^T have the same left nullspace

FALSE. For example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

Then $(1, -1)$ is in the left nullspace of A^T, but not in the left nullspace of A.

$(0, 1)$ is in the left nullspace of A, but not in the left nullspace of A^T.

c) If the row space = column space, then $A = A^T$.

FALSE. For example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq A$.

But the row space of A is $\mathbb{R}^2$, and so is the column space. (they even have the same basis, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$).

d) If $A^T = -A$, then the row space of A equals the column space.

TRUE. row space of $A = C(A^T) = C(-A)$ since $A^T = -A$.

But $C(A) = C(-A)$ since the columns of $-A$ are just the columns of A times a constant. Thus, any linear combination of columns of A can clearly be written as a linear combination of columns of $-A$ and vice versa.

So the row space of A = the column space of A in this case.