Assignment 3 (Due 9/27/12)

§ 3.1

10) a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$ IS a subspace.

because: $(0,0,0)$ is in the space

if (b, b, b_3) is in the space, $c \cdot (b, b, b_3) = (cb, cb, cb_3)$

is also (since $cb = cb$).

and if (b, b, b_3) and (c, c, c_3) are in the space, so

is their sum $(b+c, b+c, b_3+c_3)$ since $b+c = b+c$.

b) The plane of vectors with $b_1 = 1$ is not a subspace.

It is not closed under scalar multiplication or addition.

For example $(1, 0, 0)$ is in the plane, but

$2(1, 0, 0) = (2, 0, 0)$ is not.

c) The vectors with $b_1 b_2 b_3 = 0$ do not form a subspace.

$b_1 b_2 b_3 = 0$ implies that at least one of b_1, b_2, b_3 is 0.

This space is not closed under addition.

For example, $(1, 0, 1)$ and $(0, 1, 0)$ are in the space.

But their sum, $(1, 1, 1)$ isn't since $1 \cdot 1 \cdot 1 = 1 \neq 0$.

d) All linear combinations of $\vec{v} = (1, 4, 0)$ and $\vec{w} = (2, 2, 2)$

do form a subspace.

$(c_1\vec{v} + d_1\vec{w}) + (c_2\vec{v} + d_2\vec{w}) = (c_1+c_2)\vec{v} + (d_1+d_2)\vec{w}$, so

the space is closed under addition.

$c(d\vec{v} + e\vec{w}) = cd\vec{v} + ce\vec{w}$, so the space is closed under

scalar multiplication.

$0\vec{v} + 0\vec{w} = \vec{0}$, so $\vec{0}$ is in the space. ✓

e) All vectors satisfying $b_1 + b_2 + b_3 = 0$ do form a subspace.

Suppose (b_1, b_2, b_3) and (c_1, c_2, c_3) are in the space.

Then their sum $(b_1+c_1, b_2+c_2, b_3+c_3)$ is in the space

since $(b_1+c_1) + (b_2+c_2) + (b_3+c_3) = (b_1+b_2+b_3) + (c_1+c_2+c_3)$

$= 0 + 0 = 0$.

Also, the space is closed under scalar multiplication.

$c(b_1, b_2, b_3) = (cb_1, cb_2, cb_3)$ is in the space

because $cb_1 + cb_2 + cb_3 = c(b_1 + b_2 + b_3) = c \cdot 0 = 0$.

Finally, $\vec{0}$ is in the space since $0+0+0=0$.

10) f) All vectors with $b_1 \leq b_2 \leq b_3$ do <u>not</u> form a subspace.

Closure under scalar multiplication fails.

For example, $(1, 2, 3)$ is in the space.

But $(-1)(1,2,3) = (-1,-2,-3)$ is not

since $-1 \gneqq -2 \gneqq -3$.

18) a) Symmetric matrices in M form a subspace. TRUE

For example, $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ ✓

b) Skew symmetric matrices in M form a subspace. TRUE.

For example, $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ✓

c) Unsymmetric matrices in M form a subspace. FALSE.

For example,
$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{not symmetric}} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}}_{\text{symmetric}}.$$

20) a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

For which b_1, b_2, b_3 is this solvable?

Try solving with elimination:

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_1 + b_3 \end{array}\right].$$

So for there to be a solution, we need

$$b_2 - 2b_1 = 0$$
$$b_1 + b_3 = 0$$

In other words, the $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ that make the system solvable

are exactly those of the form $\begin{bmatrix} b_1 \\ 2b_1 \\ -b_1 \end{bmatrix}$.

20) b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ For which b_1, b_2, b_3 is this solvable?

Try solving with elimination:

$$\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_1 + b_3 \end{array}\right].$$

So for there to be a solution, we need $b_1 + b_3 = 0$.

In other words, there is a solution when $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is of the form $\begin{bmatrix} b_1 \\ b_2 \\ -b_1 \end{bmatrix}$.

23) If we add on extra column $\vec{b}$ to a matrix A, the column space gets larger unless $\underline{\vec{b} \text{ is a linear combination of the columns of } A}$ (i.e. $\vec{b}$ is in the column space of A).

Example where the column space gets larger:

$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. column space of $A = c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (line.)

Column space of $[A | \vec{b}] = \mathbb{R}^2$ (plane)

Example where the column space doesn't get larger:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Column space of $A = \mathbb{R}^2$

Column space of $[A \; \vec{b}] = \mathbb{R}^2$.

$A\vec{x} = \vec{b}$ is solvable exactly when the column space doesn't get larger because when $\vec{b}$ is in the column space of A $\vec{b}$ is a linear combination of the columns. This means we can find $\vec{x}$ so that $A\vec{x} = \vec{b}$ ($\vec{x}$ just tells us which linear combination).

§3.2

1) a) $A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \longrightarrow \underset{\overset{\shortparallel}{U}}{\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$

For A: pivot variables are x_1 and x_3. Free variables are x_2, x_4, x_5

b) $B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} = U.$

For B: pivot variables are x_1 and x_2

Free variable is x_3.

2) For A: $x_1 + 2x_2 + 2x_3 + 4x_4 + 6x_5 = 0$

$x_3 + 2x_4 + 3x_5 = 0.$

Special solutions: when $x_2 = 1$ and $x_4 = x_5 = 0$:

[redacted]

$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

when $x_2 = x_5 = 0$ and $x_4 = 1$: $\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

When $x_2 = x_4 = 0$ and $x_5 = 1$:

$\begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

Special solutions are $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$

For B : special solution is when $x_3 = 1$

$2x_1 + 4x_2 + 2x_3 = 0$

$4x_2 + 4x_3 = 0$

Special solution: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$

3) Every solution to $A\vec{x} = 0$ is a linear combination of special solutions.

$$\vec{x} = a\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2a \\ a \\ -2b-3c \\ b \\ c \end{bmatrix}$$

Every solution to $B\vec{x} = 0$ is a linear combination of special solutions.

$$\vec{x} = a\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ -a \\ a \end{bmatrix}.$$

The nullspace contains only $\vec{x} = \vec{0}$ when there are no free variables.

4) $$A \rightarrow U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$B \rightarrow U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R.$$

The nullspace of R = the nullspace of U. TRUE.

5) a) $$A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix}.$$

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad \text{so} \quad E_{21}A = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix} = U$$

b) $$B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad \text{so} \quad E_{21}B = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} = U$$

$$B = E_{21}^{-1}\begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}}_{U}$$

$$= L \quad U$$

21) Construct a matrix whose nullspace consists of all combinations of $(2,2,1,0)$ and $(3,1,0,1)$.

Looking at these vectors, it looks like x_3 and x_4 are free variables.

So we expect A to look like

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If $(2,2,1,0)$ and $(3,1,0,1)$ are in the nullspace, we need

$2a + 2b + c = 0$ and $3a + b + d = 0$

$2e + f = 0$ $\quad$ $e + g = 0$.

An example of numbers that satisfy this is:

$a = 1,\ b = 0,\ c = -2,\ d = -3$

$e = 1,\ f = -2,\ g = -1$.

So $A = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ fits the requirements (there are other answers).

30) a) A and A^T don't usually have the same nullspace:

For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ and $A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}$ can't have the same nullspace - they aren't even subspaces of the same R^n.

$N(A)$ is in R^2 and $N(A^T)$ is in R^3.

b) A and A^T don't usually have the same free variables:

For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ has x_3 as a free variable.

$A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ has no free variables.

30) c) If $R = \text{rref}(A)$ then in general R^T is not $\text{rref}(A^T)$.

Using the same example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$, $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} = R$.

$$R^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}, \quad \text{but} \quad \text{rref}(A^T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

§ 3.3

8) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}$

$M = \begin{bmatrix} a & b \\ c & \frac{bc}{a} \end{bmatrix}$ (need $a \neq 0$. If $a = 0$, then c must be 0 and we can fill in the blank space freely).

$B = \begin{bmatrix} 3 & 9 & -9/2 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{bmatrix}$

12) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ is invertible. (rank = 2)

$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \rightsquigarrow [1]$ is invertible. (rank = 1).

$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible (rank = 2)

19) A, B are $n \times n$ matrices with $AB = I$.

Given $\text{rank}(AB) \leq \text{rank}(A)$, show $\text{rank}(A) = n$.

$\text{rank}(AB) = \text{rank}(I) = n$, so $\text{rank}(A) \geq n$. But A is $n \times n$, so $\text{rank}(A) \leq n$.

Thus $\text{rank}(A) = n$.

So A is invertible and $AB = BA = I$.

21) Suppose A and B have the same reduced row echelon form.

a) A and B will have the same nullspace because we use the equations from the reduced row echelon form to find the nullspace.

Alternatively, if R is the common reduced row echelon form, we get

$D_1 E_1 A = R$ and $D_2 E_2 B = R$ where D_i are diagonal scaling matrices and E_i are products of elimination matrices.

Since the D's and the E's are invertible,

$$D_1 E_1 A = D_2 E_2 B \Longrightarrow A = E_1^{-1} D_1^{-1} D_2 E_2 B$$

$$\text{and } B = E_2^{-1} D_2^{-1} D_1 E_1 A.$$

So if $\vec{x} \in N(A)$, $B\vec{x} = E_2^{-1} D_2^{-1} D_1 E_1 A \vec{x} = E_2^{-1} D_2^{-1} D_1 E_1 \vec{0} = \vec{0}$ and $\vec{x} \in N(B)$. Similarly, if $\vec{x} \in N(B)$, then $\vec{x} \in N(A)$.

Therefore $N(A) = N(B)$.

A and B have the same row space because they both arrive at R by elimination, and elimination doesn't change the row space. In particular, the rows of R are linear combinations of the rows of A and of the rows of B, and because elimination is reversible, linear combinations of the rows of R give the rows of A and the rows of B. So in fact Row space(R) = Row space(A) = row space(B).

b) A equals an <u>invertible</u> matrix times B (see above).