# Assignment 2 (Due 9/20)

§2.3

7) E subtracts 7 times row 1 from row 3.

a) To invert that, you should add seven times row 1 to row 3.

b) $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}$

Check: $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}$. $E^{-1}E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

c) $EE^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

2b) Use $[A \mid \vec{b}\ \vec{b}^*]$.

$$\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

So $\left.\begin{aligned} -y &= -2 \\ -v &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} y &= 2 \\ v &= -1 \end{aligned}$

$x + 4y = 1 \Longrightarrow x + 8 = 1 \Rightarrow x = -7$

$u + 4v = 0 \Rightarrow u + -4 = 0 \rightarrow u = 4.$

So $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$

27) $[A\ \vec{b}] = \begin{bmatrix} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{bmatrix}$.

a) There is no solution if $a=b=d=0$ and $c=1$.

b) There are infinitely many solutions if $a=b=c=d=0$.

The values of $a$ ~~ab~~ and $b$ have no effect on solvability.

§2.4.

7). a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB.

TRUE.

$$A[\vec{b}_1\ \vec{b}_2\ \vec{b}_3] = [A\vec{b}_1\ \ A\vec{b}_2\ \ A\vec{b}_3].$$

If $\vec{b}_1 = \vec{b}_3$, $A\vec{b}_1 = A\vec{b}_3$.

b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB.

FALSE.

For example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}.$$

c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of ABC.

TRUE.

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}(BC) = \begin{bmatrix} \vec{a}_1 \cdot BC \\ \vec{a}_2 \cdot BC \\ \vec{a}_3 \cdot BC \end{bmatrix}$$

If $\vec{a}_1 = \vec{a}_3$, $\vec{a}_1 \cdot BC = \vec{a}_3 \cdot BC$.

d) $(AB)^2 = A^2B^2$. FALSE.

For example, set $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $(AB)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

but $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $A^2B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

11) a) $BA = 4A$.

$$B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

b) $BA = 4B$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

c) $BA$ has rows 1 and 3 of $A$ reversed and row 2 unchanged.

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

d) All rows of $BA$ are the same as row 1 of $A$.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

14) a) If $A^2$ is defined, then $A$ is necessarily square.

TRUE.

b) If $AB$ and $BA$ are defined, then $A$ and $B$ are square.

FALSE. ($A$ could be $2\times3$, $B$ could be $3\times2$).

c) If $AB$ and $BA$ are defined, then $AB$ and $BA$ are square.

TRUE.

d) If $AB = B$, then $A = I$.

FALSE. If $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A$ could be anything.

§2.5

7) If $A$ has row1 + row 2 = row3, $A$ is not invertible.

a) $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ means finding a linear combination of the columns of $A$ that gives $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

But if row 1 + row 2 = row 3, then every column of $A$ has the form $\begin{bmatrix} v_1 \\ v_2 \\ v_1+v_2 \end{bmatrix}$, and any linear combination of the columns will

look like
$$c\begin{bmatrix} v_1 \\ v_2 \\ v_1+v_2 \end{bmatrix} + d\begin{bmatrix} u_1 \\ u_2 \\ u_1+u_2 \end{bmatrix} + e\begin{bmatrix} w_1 \\ w_2 \\ w_1+w_2 \end{bmatrix} = \begin{bmatrix} cv_1 + du_1 + ew_1 \\ cv_2 + du_2 + ew_2 \\ c(v_1+v_2) + d(u_1+u_2) + e(w_1+w_2) \end{bmatrix}$$
$$= \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}.$$

Since $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not of the form $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$, there is no solution to $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

b) If $b_1 + b_2 = b_3$, there could be a solution to $A\vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

c) In elimination, row 3 becomes all zeros.

25) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{E_{12}} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{E_{21}} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 3 & 1 & -1 & 2 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{E_{31}} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 3 & 1 & -1 & 2 & 0 \\ 0 & 2 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{E_{23}} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 2 & 2 & -1 & 1 & 1 \end{array}\right] \xrightarrow{E_{32}} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 4 & -1 & -1 & 3 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{array}\right] \longrightarrow$$

25 cont.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{array}\right].$$

$$A^{-1} = \begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & -1/2 & 1/2 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & -3/2 & 1/2 & 1 & 0 \\ 0 & -3/2 & 3/2 & 1/2 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & -3/2 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

Not invertible! (No 3RD pivot).

29) a) A $4 \times 4$ matrix with a row of 0's is not invertible. TRUE.

b) Every matrix with 1's down the main diagonal is invertible. FALSE.

For example, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible.

c) If $A$ is invertible, then $A^{-1}$ and $A^2$ are invertible. TRUE.

$(A^{-1})^{-1} = A$ and $(A^2)^{-1} = (A^{-1})^2$.

40) $A = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & -ab & 0 & 1 & a & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

40 cont.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -ab & 0 & 1 & a & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -abc & 1 & a & ab & 0 \\ 0 & 1 & 0 & -bc & 0 & 1 & b & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & a & ab & abc \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

§ 2.6

7) $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_{21}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix}$

$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ and $E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}$.

$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ and $E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

13) $A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$

$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$

$E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ a & b & c & d \end{bmatrix}$

$E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$ and $E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$

$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & b-a & c-a & d-a \end{bmatrix}$

$E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$ and $E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$

$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ and $E_{43}E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$.

So $A = LU = E_{21}^{-1}E_{31}^{-1}E_{41}^{-1}E_{32}^{-1}E_{42}^{-1}E_{43}^{-1} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b-a & 0 & 0 \\ 0 & 0 & c-b & 0 \\ 0 & 0 & 0 & d-c \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

16) $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. $\quad L\vec{c} = \vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

$$\begin{aligned} c_1 &= 4 \\ c_1 + c_2 &= 5 \longrightarrow c_2 = 1 \\ c_1 + c_2 + c_3 &= 6 \longrightarrow c_3 = 1. \end{aligned} \qquad \text{So } \vec{c} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad U\vec{x} = \vec{c}$.

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \longrightarrow x_1 = 3. \\ x_2 + x_3 &= 1 \longrightarrow x_2 = 0 \\ x_3 &= 1 \end{aligned}$$

So $\vec{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

§2.7

4) $A^2 = 0$ is possible: For example $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$A^TA$ is not possible: If $A = [\vec{v}_1\ \vec{v}_2 \dots \vec{v}_n]$, $A^TA = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} [\vec{v}_1 \dots \vec{v}_n]$

So $A^TA = \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \vec{v}_1 \cdot v_3 \cdots & \vec{v}_1 \cdot \vec{v}_n \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 & \cdots & \vec{v}_2 \cdot \vec{v}_n \\ \vdots & & & \vdots \\ \vec{v}_n \cdot \vec{v}_1 & \cdots & & \vec{v}_n \cdot \vec{v}_n \end{bmatrix}$

If $A^TA = 0$, in particular $\vec{v}_i \cdot \vec{v}_i = \|\vec{v}_i\|^2 = 0$ for all $i$.
But then $\vec{v}_i = \vec{0}$.

So $A^TA = 0$ only when $\vec{v}_i = \vec{0}$, and $A = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & & \vdots \\ 0 & & \\ \vdots & & \\ 0 & & 0 \end{bmatrix}$

9) If $P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,

$$P_1 P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

but $P_2 P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$

If $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $P_3 P_4 = P_4 P_3$.

24) $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix}.$

Set $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, so $PA = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 8 \\ 0 & 1 & 2 \end{bmatrix}.$

Then $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}$ and $E_{32}PA = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 8 \\ 0 & 0 & -2/3 \end{bmatrix}$

So

$$\underset{P}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}\underset{A}{\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 8 \\ 0 & 0 & -2/3 \end{bmatrix}}$$

OR: $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix}$

Then $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_{21}A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}$

Set $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, so $PE_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$

Then $A = E_{21}^{-1} P^{-1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

40) Suppose $Q^T = Q^{-1}$

a) Say $Q = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$. Then $Q^T = \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_n \end{bmatrix}$ and

$$Q^T Q = \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \cdots & \vec{v}_1 \cdot \vec{v}_n \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 & \cdots & \vdots \\ \vdots & & & \\ \vec{v}_n \cdot \vec{v}_1 & \cdots & & \vec{v}_n \cdot \vec{v}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & & 1 \end{bmatrix}$$

So in particular, $\vec{v}_i \cdot \vec{v}_i = \|\vec{v}_i\|^2 = 1$ for all $i$.

So $\|\vec{v}_i\|^2 = 1$ and the columns of $Q$ are unit vectors.

b) We also see that if $i \neq j$, $\vec{v}_i \cdot \vec{v}_j = 0$ (That is, $\vec{v}_i^T \vec{v}_j = 0$)

and $\vec{v}_i$ and $\vec{v}_j$ are perpendicular.

c) $\vec{v}_1$ is a unit vector $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

and $\vec{v}_2$ is perpendicular to $\vec{v}_1$. If $\vec{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$, we need $a\cos\theta + b\sin\theta = 0$

and $a^2 + b^2 = 1$.

So $\vec{v}_2 = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$ works.

$$Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$