Assignment 12 : Due 12/6

§ 7.3

1) a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$$

Eigenvalues: $\det\begin{bmatrix} 10-\lambda & 20 \\ 20 & 40-\lambda \end{bmatrix} = 400 - 50\lambda + \lambda^2 - 400 = \lambda(\lambda - 50)$

$\lambda_1 = 50, \quad \lambda_2 = 0$

Eigenvectors: $\lambda_1 = 50$: $\begin{bmatrix} -40 & 20 \\ 20 & -10 \end{bmatrix}\vec{x} = 0 \rightsquigarrow \vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = 0$: $\begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}\vec{x} = 0 \rightsquigarrow \vec{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\sigma_1 = \sqrt{50} = 5\sqrt{2}$

b) $AA^T = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}$

Eigenvalues: $\det\begin{bmatrix} 5-\lambda & 15 \\ 15 & 45-\lambda \end{bmatrix} = 225 - 50\lambda + \lambda^2 - 225 = \lambda(\lambda - 50)$

$\lambda_1 = 50, \quad \lambda_2 = 0.$

Eigenvectors: $\lambda_1 = 50$: $\begin{bmatrix} -45 & 15 \\ 15 & 45 \end{bmatrix}\vec{x} = 0 \rightsquigarrow \vec{u}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda_2 = 0$: $\begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}\vec{x} = 0 \rightsquigarrow \vec{u}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

c) $A\vec{v}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 5 \\ 15 \end{bmatrix} = \sqrt{5}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{5\sqrt{2}}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \sigma_1\vec{u}_1$ ✓

$$A = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

2) a) Basis for $C(A)$: $\frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}$

Basis for $N(A^T)$: $\frac{1}{\sqrt{10}}\begin{bmatrix}3\\-1\end{bmatrix}$

Basis for $C(A^T)$: $\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$

Basis for $N(A)$: $\frac{1}{\sqrt{5}}\begin{bmatrix}2\\-1\end{bmatrix}$

b) Matrices with the same four subspaces as A will all be of the form

$$U\begin{bmatrix}\sigma' & 0\\ 0 & 0\end{bmatrix}V^T$$ where

the U and V remain the same (and only σ changes from the decomposition of A).

This actually means only multiples of A have the same subspaces in this case.

3) $$Q = UV^T = \frac{1}{\sqrt{10}}\cdot\frac{1}{\sqrt{5}}\begin{bmatrix}1 & 3\\ 3 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix} = \frac{1}{5\sqrt{2}}\begin{bmatrix}7 & -1\\ 1 & 7\end{bmatrix}$$

$$H = V\Sigma V^T = \frac{1}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}5\sqrt{2} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}5\sqrt{2} & 0\\ 10\sqrt{2} & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$$

$$= \begin{bmatrix}\sqrt{2} & 2\sqrt{2}\\ 2\sqrt{2} & 4\sqrt{2}\end{bmatrix}$$

$$QH = \frac{1}{5\sqrt{2}}\cdot\sqrt{2}\begin{bmatrix}7 & -1\\ 1 & 7\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} = \frac{1}{5}\begin{bmatrix}5 & 10\\ 15 & 30\end{bmatrix} = \begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix} = A \checkmark$$

H is only semidefinite because the diagonal entries of Σ give its eigenvalues and Σ has one diagonal entry equal to 0.

$$H^2 = (\sqrt{2})^2\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} = 2\begin{bmatrix}5 & 10\\ 10 & 20\end{bmatrix} = \begin{bmatrix}10 & 20\\ 20 & 40\end{bmatrix} = A^TA \checkmark$$

4) $$A^+ = V\Sigma^+U^T = \frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}1/\sqrt{50} & 0\\ 0 & 0\end{bmatrix}\frac{1}{\sqrt{10}}\begin{bmatrix}1 & 3\\ 3 & -1\end{bmatrix}$$

$$= \frac{1}{5\sqrt{2}}\begin{bmatrix}\frac{1}{5\sqrt{2}} & 0\\ \frac{2}{5\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}1 & 3\\ 3 & -1\end{bmatrix} = \frac{1}{5\sqrt{2}}\begin{bmatrix}\frac{1}{5\sqrt{2}} & \frac{3}{5\sqrt{2}}\\ \frac{2}{5\sqrt{2}} & \frac{6}{5\sqrt{2}}\end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix}1 & 3\\ 2 & 6\end{bmatrix}$$

$C(A^+) = C(A^T)$

$N(A^{+T}) = N(A)$

$C(A^{+T}) = C(A)$

$N(A^+) = N(A^T)$

18) $A = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \underset{U}{\begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} 5 \\ 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 1 \end{bmatrix}}$ $\quad \vec{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

$$A^+ = V\Sigma^+ U^T = [1] \begin{bmatrix} \frac{1}{5} & 0 \end{bmatrix} \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix} = \begin{bmatrix} \frac{3}{25} & \frac{4}{25} \end{bmatrix}$$

$$A^+A = \begin{bmatrix} \frac{3}{25} & \frac{4}{25} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [1]$$

$$AA^+ = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} \frac{3}{25} & \frac{4}{25} \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

$$\vec{x}^+ = A^+\vec{b} = \begin{bmatrix} \frac{3}{25} & \frac{4}{25} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [1]$$

$$\vec{x}^+ = A^+\vec{b} = \begin{bmatrix} \frac{3}{25} & \frac{4}{25} \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = [0]$$

§10.2

6) a) If A is real, $A + iI$ is invertible FALSE

example: If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A + iI = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$ has determinant 0 and is singular

b) If A is Hermitian, then $A + iI$ is invertible. TRUE

The eigenvalues of $A + iI$ are $\lambda_1 + i, \lambda_2 + i, \dots, \lambda_n + i$ where $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of A.

(Since $A\vec{x}_k = \lambda_k \vec{x}_k \Rightarrow (A + iI)\vec{x}_k = (\lambda_k + i)\vec{x}_k$.)

A Hermitian means that all its eigenvalues are real, so

$$\lambda_k + i \neq 0 \text{ for any } k.$$

Since $A + iI$ has no eigenvalues equal to 0, its determinant is nonzero and $A + iI$ is invertible.

c) If A is unitary, $A + iI$ is invertible FALSE.

example: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is unitary, but $\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$ is singular.

14) $A = \begin{bmatrix} 0 & 1-i \\ 1+i & 1 \end{bmatrix}$

Eigenvalues: $\det(A-\lambda I) = \det \begin{bmatrix} -\lambda & 1-i \\ 1+i & 1-\lambda \end{bmatrix} : \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$

$\lambda_1 = 2, \quad \lambda_2 = -1$

Eigenvectors: $\lambda_1 = 2$: $\begin{bmatrix} -2 & 1-i \\ 1+i & -1 \end{bmatrix} \vec{x} = 0 \rightsquigarrow \vec{x} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

$\lambda_2 = -1$: $\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix} \vec{x} = 0 \rightsquigarrow \vec{x} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}.$$

15) $K = \begin{bmatrix} 0 & -1+i \\ 1+i & i \end{bmatrix}$ Eigenvalues: $\det(A-\lambda I) = \det \begin{bmatrix} -\lambda & -1+i \\ 1+i & i-\lambda \end{bmatrix} = \lambda^2 - i\lambda + 2$

$$\lambda = \frac{i \pm \sqrt{-1-8}}{2} = \frac{i \pm 3i}{2} = 2i, \ -i$$

$\lambda_1 = 2i, \quad \lambda_2 = -i$

Eigenvectors: $\lambda_1 = 2i$ $\begin{bmatrix} -2i & -1+i \\ 1+i & -i \end{bmatrix} \vec{x} = 0 \rightsquigarrow \frac{1}{\sqrt{3}} \begin{bmatrix} i \\ 1-i \end{bmatrix}$

$\lambda_2 = -i$ $\begin{bmatrix} i & -1+i \\ 1+i & 2i \end{bmatrix} \vec{x} = 0 \rightsquigarrow \frac{1}{\sqrt{3}} \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} i & 1+i \\ 1-i & -1 \end{bmatrix} \begin{bmatrix} 2i & 0 \\ 0 & -i \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} -i & 1+i \\ 1-i & -1 \end{bmatrix}$$

All λ's are purely imaginary.

16) $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Eigenvalues: $\det(Q-\lambda I) = \det\begin{bmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{bmatrix} = \lambda^2 - 2\cos\theta\,\lambda + \cos^2\theta + \sin^2\theta$

$= \lambda^2 - 2\cos\theta\,\lambda + 1.$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta$$

$\lambda_1 = e^{i\theta}$, $\lambda_2 = e^{-i\theta}$

Eigenvectors: $\lambda_1 = e^{i\theta} = \cos\theta + i\sin\theta$: $\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}\vec{x} = 0 \rightsquigarrow \vec{x}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda_2 = e^{-i\theta} = \cos\theta - i\sin\theta$: $\begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}\vec{x} = 0 \rightsquigarrow \vec{x}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}$$

All λ's are complex (with length 1)

17) $V = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$ Eigenvalues: $\det(V-\lambda I) = \det\begin{bmatrix} \frac{1}{\sqrt{3}}-\lambda & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & -\frac{1}{\sqrt{3}}-\lambda \end{bmatrix}$

$= -\frac{1}{3} + \lambda^2 - \frac{2}{3} = \lambda^2 - 1 = (\lambda+1)(\lambda-1).$

Eigenvectors: $\lambda_1 = 1$: $\begin{bmatrix} \frac{1-\sqrt{3}}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{-1-\sqrt{3}}{\sqrt{3}} \end{bmatrix}\vec{x} = 0 \rightsquigarrow (1-\sqrt{3})x_1 + (1-i) = 0$

$x_1 = \frac{-1+i}{1-\sqrt{3}}$

$\vec{x}_1 = \frac{1}{\sqrt{1+1+(1-\sqrt{3})^2}}\begin{bmatrix} -1+i \\ 1-\sqrt{3} \end{bmatrix}$

$= \frac{1}{\sqrt{6-2\sqrt{3}}}\begin{bmatrix} -1+i \\ 1-\sqrt{3} \end{bmatrix}$

$\lambda_2 = -1$ $\begin{bmatrix} \frac{1+\sqrt{3}}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{-1+\sqrt{3}}{\sqrt{3}} \end{bmatrix}\vec{x} = 0 \rightsquigarrow (1+\sqrt{3})x_1 + 1 - i = 0$

$x_1 = \frac{-(1-i)}{1+\sqrt{3}}$

$\vec{x}_2 = \frac{1}{\sqrt{6+2\sqrt{3}}}\begin{bmatrix} -1+i \\ 1+\sqrt{3} \end{bmatrix}$

$$V = \frac{1}{\sqrt{6+2\sqrt{3}}}\begin{bmatrix} -1+i & -1+i \\ 1-\sqrt{3} & 1+\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{\sqrt{6+2\sqrt{3}}}\begin{bmatrix} -1+i & -1+i \\ 1-\sqrt{3} & 1+\sqrt{3} \end{bmatrix}$$

The λ are numbers with absolute value 1 (and real).

30) $A = U\Lambda U^{-1} = U\Lambda U^H$.

Then $AA^H = (U\Lambda U^H)(U\Lambda U^H)^H = U\Lambda(U^H U)\Lambda^H U^H$

$= U\Lambda\Lambda^H U^H$

But $\Lambda\Lambda^H = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}\begin{bmatrix} \bar{\lambda}_1 & & & \\ & \bar{\lambda}_2 & & \\ & & \ddots & \\ & & & \bar{\lambda}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\bar{\lambda}_1 & & & \\ & \lambda_2\bar{\lambda}_2 & & \\ & & \ddots & \\ & & & \lambda_n\bar{\lambda}_n \end{bmatrix}$

$= \begin{bmatrix} \bar{\lambda}_1 & & & \\ & \bar{\lambda}_2 & & \\ & & \ddots & \\ & & & \bar{\lambda}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \Lambda^H\Lambda$

So $AA^H = U\Lambda\Lambda^H U^H = U\Lambda^H\Lambda U^H = (U\Lambda^H U^H)(U\Lambda U^H)$

$= A^H A$ ✓

Example: choose $U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $U^H = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$\Lambda = \begin{bmatrix} 2i & \\ & 1+i \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2i & \\ & 1+i \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1+i \\ -2i & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1+i & 0 \\ 0 & 2i \end{bmatrix}$ is normal.

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6) $F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$, $F_4^2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$

$$F_4^4 = 4^2\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^2 = 16\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= 16\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

11) $$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & i & & \\ & & i^2 & \\ & & & i^3 \end{bmatrix}$$

$$\lambda_1 = 1, \quad \lambda_2 = i, \quad \lambda_3 = i^2, \quad \lambda_4 = i^3$$

13) a) $$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_0 + c_1 + c_2 + c_3 \\ c_0 + c_1 + c_2 + c_3 \\ c_0 + c_1 + c_2 + c_3 \\ c_0 + c_1 + c_2 + c_3 \end{bmatrix} \qquad e_1 = c_0 + c_1 + c_2 + c_3.$$

b) $$C\begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix} = \begin{bmatrix} c_0 + c_1 i + c_2 i^2 + c_3 i^3 \\ c_3 + c_0 i + c_1 i^2 + c_2 i^3 \\ c_2 + c_3 i + c_0 i^2 + c_1 i^3 \\ c_1 + c_2 i + c_3 i^2 + c_0 i^3 \end{bmatrix}. \qquad e_2 = c_0 + c_1 i + c_2 i^2 + c_3 i^3$$

b) $P = F\Lambda F^{-1}$

$$C = F(\underbrace{c_0 I + c_1 \Lambda + c_2 \Lambda^2 + c_3 \Lambda^3}_{E})F^{-1}.$$

E contains the eigenvalues of C.

14) $E = 2I - \Lambda - \Lambda^3$ with the eigenvalues of P in Λ.

From 11, the eigenvalues of P are $1, i, i^2, i^3$.

so $$E = 2I - \begin{bmatrix} 1 & & & \\ & i & & \\ & & i^2 & \\ & & & i^3 \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & i^3 & & \\ & & i^6 & \\ & & & i^9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & & & \\ & 2-i+i & & \\ & & 2-(-1)-(-1) & \\ & & & 2-(-i)-(i) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & & & \\ & 2 & & \\ & & 4 & \\ & & & 2 \end{bmatrix}$$

Eigenvalues of C are $\lambda_1 = 0$, $\lambda_2 = 4$, $\lambda_3 = \lambda_4 = 2$.