Assignment 10: Due 11/20

§8.3

1) $A = \begin{bmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{bmatrix}$

Eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = (0.9 + 0.85) - 1 = 0.75$

Steady state eigenvector:

$$A - I = \begin{bmatrix} -0.1 & 0.15 \\ 0.1 & -0.15 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \rightsquigarrow \text{Rescale to get steady state} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$$

2) $A = \begin{bmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{bmatrix}$

$\lambda = 0.75$ eigenvector: $A - 0.75 I = \begin{bmatrix} 0.15 & 0.15 \\ 0.1 & 0.1 \end{bmatrix}$

$$\vec{x}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3/5 & 1/2 \\ 2/5 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix} (-2) \begin{bmatrix} -1/2 & -1/2 \\ -\frac{2}{5} & 3/5 \end{bmatrix}$$

$$\lim_{k \to \infty} A^k = \begin{bmatrix} 3/5 & 1/2 \\ 2/5 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (-2) \begin{bmatrix} -1/2 & -1/2 \\ -2/5 & 3/5 \end{bmatrix}$$

$$= (-2) \begin{bmatrix} 3/5 & 0 \\ 2/5 & 0 \end{bmatrix} \begin{bmatrix} -1/2 & -1/2 \\ -2/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{bmatrix}$$

3) $A = \begin{bmatrix} 1 & 0.2 \\ 0 & 0.8 \end{bmatrix}$ $\lambda_1 = 1, \ \lambda_2 = 0.8$

Steady state: $A - I = \begin{bmatrix} 0 & 0.2 \\ 0 & -0.2 \end{bmatrix} \rightsquigarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ = steady state.

$A = \begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix}$ $\lambda_1 = 1, \ \lambda_2 = -0.8$

Steady state: $A - I = \begin{bmatrix} -0.8 & 1 \\ 0.8 & -1 \end{bmatrix} \rightsquigarrow \vec{x}_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

Steady state: $\begin{bmatrix} 5/9 \\ 4/9 \end{bmatrix}$

$A = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$ $\lambda_1 = 1$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1/2-\lambda & \frac{1}{4} & 1/4 \\ 1/4 & 1/2-\lambda & 1/4 \\ 1/4 & 1/4 & 1/2-\lambda \end{bmatrix} = \left(\tfrac{1}{2} - \lambda\right)^3 + \frac{2}{64} - \frac{3}{16}\left(1/2 - \lambda\right)$$

$$= 1/8 - 3/4\,\lambda + 3/2\,\lambda^2 - \lambda^3 + \frac{1}{32} - \frac{3}{32} + \frac{3}{16}\lambda$$

$$= 1/16 - 9/16\,\lambda + 3/2\,\lambda^2 - \lambda^3$$

To find zeros, we can multiply through by 16:

$$-16\lambda^3 + 24\lambda^2 - 9\lambda + 1 = 0$$

$$= (-\lambda + 1)(16\lambda^2 - 8\lambda + 1)$$

$$= (-\lambda + 1)(4\lambda - 1)(4\lambda - 1)$$

Eigenvalues: $\lambda_1 = 1$

$\lambda_2 = \lambda_3 = 1/4$

Steady state $A - I = \begin{bmatrix} -1/2 & 1/4 & 1/4 \\ 1/4 & -1/2 & 1/4 \\ 1/4 & 1/4 & -1/2 \end{bmatrix}$ $\vec{x}_1 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ (note that all rows of $A - I$ sum to 0)

5) $\begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_{k+1} = \begin{bmatrix} 0.98 & 0 & 0 \\ 0.02 & 0.97 & 0 \\ 0 & 0.03 & 1 \end{bmatrix} \begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_k$.

Steady state corresponds to $\lambda_1 = 1$

$$A - I = \begin{bmatrix} -0.02 & 0 & 0 \\ 0.02 & -0.03 & 0 \\ 0 & 0.03 & 0 \end{bmatrix}$$

$(A-I)\vec{x} = 0 \rightsquigarrow \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the steady state (everyone dies).

12) $B = A - I = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix}$.

$A - I$ has $\lambda = 0$ because A has $\lambda = 1$, so there is some $\vec{x}$ (not the zero vector) with $A\vec{x} = \vec{x}$. Then $(A-I)\vec{x} = A\vec{x} - I\vec{x} = \vec{x} - \vec{x} = 0$, so $\vec{x}$ is an eigenvector for $A - I$ with $\lambda = 0$.

(Alternatively, in this case it is easy to see that $A - I$ is singular. Or since we know a markov matrix always has $\lambda_1 = 1$, $A - I$ will always be singular. There are many possible correct explanations).

General solution: $\vec{u}(t) = Ce^{0t}\vec{x}_1 + De^{-0.5t}\vec{x}_2$ ($\lambda_2 = -0.5$ since $\lambda_1 + \lambda_2 = -0.5 =$ trace)

So as $t \to \infty$, $\vec{u}(t) \to C\vec{x}_1$

To find $\vec{x}_1$: $B - 0I = B$

$B\vec{x} = 0$ $\vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ Rescale: $\vec{x}_1 = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$.

So $\vec{u}(t) \to C\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$ as $t \to \infty$.

16) $A = \begin{bmatrix} 0.4 & 0.2 & 0.3 \\ 0.2 & 0.4 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}$ has determinant 0 (given).

A is Markov, so $\lambda_1 = 1$

A is singular, so $\lambda_2 = 0$

$\lambda_1 + \lambda_2 + \lambda_3 = 0.4 + 0.4 + 0.4 = 1.2$, so $\lambda_3 = 0.2$.

Steady state vector

$$A - I = \begin{bmatrix} -0.6 & 0.2 & 0.3 \\ 0.2 & -0.6 & 0.3 \\ 0.4 & 0.4 & -0.6 \end{bmatrix} \to \begin{bmatrix} -6 & 2 & 3 \\ 2 & -6 & 3 \\ 4 & 4 & -6 \end{bmatrix} \to \begin{bmatrix} 2 & -6 & 3 \\ 0 & -16 & 12 \\ 0 & 16 & -12 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & -6 & 3 \\ 0 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} \overset{\text{Rescale}}{\rightsquigarrow} \vec{x}_1 = \begin{bmatrix} 3/10 \\ 3/10 \\ 4/10 \end{bmatrix} = \text{steady state.}$$

As $k \to \infty$, $A^k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix}$.

As $k \to \infty$, $A^k \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} \to 100 \cdot \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \\ 40 \end{bmatrix}$

Handout

1) $A = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 0 & 0 \end{bmatrix}$

a) A is Markov since all entries are nonnegative and columns add to 1

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$$
$$\frac{1}{2} + \frac{1}{2} + 0 = 1$$
$$\frac{1}{4} + \frac{3}{4} + 0 = 1$$

$$A^2 = \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 0 & 0 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 5 & 6 & 7 \\ 9 & 6 & 7 \\ 2 & 4 & 2 \end{bmatrix}$$ is positive.

b) Since A^2 is positive, A is a regular Markov matrix and $\lim_{k\to\infty} A^k = A^\infty$ has all columns equal to the steady state vector.

$$A - I = \begin{bmatrix} -3/4 & 1/2 & 1/4 \\ 1/4 & -1/2 & 3/4 \\ 1/2 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 3 \\ 2 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & -4 & 10 \\ 0 & 4 & -10 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 3 \\ 0 & -4 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} \overset{\text{Rescale}}{\rightsquigarrow} \vec{x}_1 = \begin{bmatrix} 4/11 \\ 5/11 \\ 2/11 \end{bmatrix}$$

$$A^\infty = \lim_{k\to\infty} A^k = \frac{1}{11}\begin{bmatrix} 4 & 4 & 4 \\ 5 & 5 & 5 \\ 2 & 2 & 2 \end{bmatrix}$$

2)

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix}$$

$$A \begin{bmatrix} JPM \\ GS \\ MS \end{bmatrix}_k = \begin{bmatrix} JPM \\ GS \\ MS \end{bmatrix}_{k+1} \quad \text{Initially, } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ (in billions)}$$

After 1000 days, the distribution is $A^{1000} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Diagonalize A to compute A^{1000}.

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 0$ (A is singular, row 1 − row 2 − row 3 = (0, 0, 0))
$\lambda_3 = \frac{1}{2}$ since $\lambda_1 + \lambda_2 + \lambda_3 = \frac{3}{2}$.

Eigenvectors:

For $\lambda_1 = 1$ $(A - I)x = 0$.

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & 0 \\ \frac{1}{4} & 0 & -\frac{1}{2} \end{bmatrix} \vec{x} = 0 \longrightarrow \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \rightarrow \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 0$. $A\vec{x} = 0$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda_3 = \frac{1}{2}$ $(A - \frac{1}{2}I)\vec{x} = 0$

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

So $A = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$

2) cont.

$$\left[\begin{array}{ccc|ccc} 2 & -2 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 2 & -1 & -\frac{1}{2} & 1 & 0 \\ 0 & 2 & 1 & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 2 & -1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

So $A^{1000} = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\frac{1}{2})^{1000} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 2 \end{bmatrix} (\frac{1}{4})$

and $A^{1000} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\frac{1}{2})^{1000} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$= \frac{1}{4} \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\frac{1}{2})^{1000} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 \\ 3/4 \\ 3/4 \end{bmatrix}$$

After 1000 days, 1.5 billion is in the JPM account

(Alternatively, see that $JPM_{new} = \frac{1}{2} JPM_{old} + \frac{1}{2} GS_{old} + \frac{1}{2} MS_{old}$
$= \frac{1}{2} (Total_{old})$

Since the total amount of money is not changing, JPM always has half the total, or 1.5 billion after the first day)

3) $\begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix} \begin{bmatrix} \text{Providence} \\ \text{Newport} \end{bmatrix}_k = \begin{bmatrix} \text{Providence} \\ \text{Newport} \end{bmatrix}_{k+1}$

Initially, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

In the long run, we'll get to the steady state.

$\lambda = 1$: $\begin{bmatrix} -2/3 & 1/2 \\ 2/3 & -1/2 \end{bmatrix} \vec{x} = 0 \rightsquigarrow \vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \xrightarrow{\text{Rescale}} \vec{x}_1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$

In the long run, $3/7$ of cars will be in Providence.