Math 520 Practice Problems for the Final

1. True or False: If A is a (square) diagonalizable matrix, then there is a matrix B so that $B^2 = A$. Explain. Let $A = \begin{bmatrix} 1 & -4 & 2 \\ 3 & -4 & 0 \\ 3 & -1 & -3 \end{bmatrix}$. Find B so that $B^2 = A$.

2. More T/F. If true, why? If false, provide a counterexample, and think about what changes would make the statement true.

   (a) The vectors $(2i, 2+3i)$ and $(2i-2, -1+5i)$ are linearly independent.

   (b) If A is Markov and A^∞ exists, then A has exactly one eigenvalue λ (counted with algebraic multiplicity) with $|\lambda| = 1$.

   (c) If $A^T = -A$, then all eigenvalues of A are purely imaginary ($= bi, b \in \mathbb{R}$).

   (d) For any matrix A, $N(A) = N(A^T A)$.

   (e) For any matrix A, $\text{rank}(A) = \text{rank}(A^T A)$.

3. Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -2 & 8 \end{bmatrix}$. Compute A^+ and $(AA^+)^2$.

4. Let $A = \begin{bmatrix} 0 & s \\ 1 & 1-s \end{bmatrix}$. For which s with $0 \leq s \leq 1$ does $A^\infty = \lim_{k\to\infty} A^k$ exist?

5. Find a basis for the orthogonal complement of the one-dimensional subspace of $\mathbb{C}^3$ spanned by $(1+i, 1, 2i)$.

6. Find a symmetric matrix A so that

$$\mathbf{x}^T A\mathbf{x} = 4\left(\frac{x_1}{\sqrt{14}} + \frac{2x_2}{\sqrt{14}} + \frac{3x_3}{\sqrt{14}}\right)^2 + 2\left(\frac{3x_1}{\sqrt{10}} - \frac{x_3}{\sqrt{10}}\right)^2.$$

7. Let $r(t)$ be the number of robins at time t, and let $w(t)$ be the number of worms at time t. Assume that the robins and worms are governed by the relationship

$$r' = r + 2w \qquad\qquad w' = -3r + 6w.$$

   Initially, there are 160 robins and 210 worms. Compute the limit

$$\lim_{t\to\infty} \frac{r(t)}{w(t)}.$$

8. The Jibonacci numbers z_k are defined by the formula

$$z_{k+2} = 3z_{k+1} - 2z_k$$

   and $z_0 = 0$, $z_1 = 1$. Find z_{100}.

9. Find the intersection of the two spaces in $\mathbb{R}^4$:

$$X = \{(0,1,1,0) + a(-3,1,0,0) + b(1,0,1,0) + c(0,0,0,1)\}$$
$$Y = \{(1,2,-1,0) + d(1,0,-2,0) + e(0,1,1,0) + f(0,0,2,1)\}$$

(Here, a, b, c, d, e, and f run over all real numbers.)

10. For the following matrices A, B and C which matrix decompositions exist? Do not compute them.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Which of the following exist?

- LU
- QR
- $S\Lambda S^{-1}$
- $Q\Lambda Q^T$
- QH
- $R^T R$
- SVD