THE GOLDEN RATIO: ALL THE COOL STUFF YOUR MOTHER NEVER TOLD YOU

LI-MEI LIM

1. Introduction

A lot of people learn that the golden ratio makes the "most beautiful" rectangles. They might also learn that the golden ratio is sometimes called φ and that

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

But why does it come up in nature so much? It's not that nature "wants" to be beautiful, and thus chooses the "most beautiful" number. There's an actual reason!

To understand this reason, we first have to learn about *continued fractions*. Once we compute the continued fraction for φ, we'll see that φ is especially irrational (and explain what that even means). Then we'll see why nature should want to be as irrational as possible.

2. Background: Continued Fractions

A continued fraction is an expression of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}}},$$

where the a_i are positive integers.

We can write any rational number in this form without too much trouble by repeatedly using *division with remainder*. This is also known as Euclid's algorithm for finding the GCD (greatest common divisor). Here's an example. Say we want to write $\frac{29}{11}$ as a continued fraction. Start with division with remainder.

$$\begin{aligned} 29 &= 2 \cdot 11 + 7 \\ 11 &= 1 \cdot 7 + 4 \\ 7 &= 1 \cdot 4 + 3 \\ 4 &= 1 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 + 0 \end{aligned}$$

From this, we see that we can write $\frac{29}{11}$ as $2 + \frac{7}{11}$. And working our way through the equations, we get

$$\frac{29}{11} = 2+\frac{7}{11} = 2+\frac{1}{\frac{11}{7}} = 2+\cfrac{1}{1+\cfrac{4}{7}} = 2+\cfrac{1}{1+\frac{1}{\frac{7}{4}}} = 2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{3}{4}}} = 2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{\frac{4}{3}}}} = 2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3}}}}.$$

Notice that the a_i's are just the coefficients 2, 1, 1, 1, 3 from Euclid's Algorithm!

Now, the "partial fractions"[1] approximate our fraction. By partial fractions, I mean we truncate the continued fraction and omit some of the lower parts. These "partial fractions" are called *convergents*. So in our example, the convergents are

$$2, \quad 2+\frac{1}{1} = 3, \quad 2+\cfrac{1}{1+\cfrac{1}{1}} = \frac{5}{2}, \quad 2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}} = \frac{8}{3}, \quad 2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3}}}} = \frac{29}{11}$$

These fractions *converge to* our number $\frac{29}{11}$, meaning they get closer and closer to $\frac{29}{11}$. Now it's your turn to practice.

Exercise 1. *Find the continued fraction decompositions for:*

$$\frac{72}{25}, \ \frac{67}{23}, \ \frac{158}{57}, \ \text{and} \ \frac{4001}{2689}.$$

A note on notation: since the numerators in a continued fraction are always 1, we really only have to pay attention to the non-numerators. In the case of $\frac{29}{11}$, those numbers were 2,1,1,1,3. To save space, we'll write

$$\frac{29}{11} = [2, 1, 1, 1, 3]$$

to mean the continued fraction with a_i's equal to 2,1,1,1,3.

3. Continued Fractions of Irrational Numbers?!

In the end, we want to understand φ, the golden ratio. This number is irrational, so we can't find its continued fraction the way we did for rational numbers, like $\frac{29}{11}$. It turns out that irrational numbers involving square roots can be written as *infinite repeating continued fractions.*

Let's look at the example of $\sqrt{2}$. If we want to express $\sqrt{2}$ as a continued fraction, we need

$$\sqrt{2} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}}}.$$

Note that $\sqrt{2} - a_1$ is less than 1, so a_1 should be the biggest integer less than $\sqrt{2}$. In other words, since $1 < \sqrt{2} < 2$, $a_1 = 1$.

[1] No, not *those* partial fractions. Not the ones from calculus.

Our computation now goes like this:

$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{\left(\frac{1}{\sqrt{2}-1}\right)}.$$

Now we have to remember how to rationalize the denominator. For the fraction in parentheses above, we multiply numerator and denominator by $\sqrt{2}+1$ to get

$$\sqrt{2} = 1 + \frac{1}{\frac{\sqrt{2}+1}{2-1}} = 1 + \frac{1}{1+\sqrt{2}}.$$

So we have a formula for $\sqrt{2}$ in terms of itself! We can plug our formula into itself to find

$$\sqrt{2} = 1 + \cfrac{1}{1 + 1 + \cfrac{1}{1+\sqrt{2}}} = 1 + \cfrac{1}{2 + \cfrac{1}{1+\sqrt{2}}}.$$

If we repeat the process, we get that $\sqrt{2} = [1, \bar{2}]$, where the bar over the 2 means that it repeats (like with repeating decimals).

Your turn! (Note: these problems are a little trickier. They will require that you repeat the process described for finding a_1 for $\sqrt{2}$, and you may need to decide what the biggest integer less than something like $\frac{a\sqrt{d}+b}{c}$ is.)

Exercise 2. *Find continued fraction expansions for $\sqrt{3}$, $\sqrt{7}$, and $\varphi = \frac{1+\sqrt{5}}{2}$.*

Just like with rational numbers, we can use the convergents (those partial fractions) to approximate the continued fractions. This is useful because irrational numbers are much harder to get a grip on. But with convergents, we find rational numbers (fractions) that are close to our irrational number.

Exercise 3. *Compute the first few convergents for the continued fraction of φ. Notice anything cool?*

4. The Golden Ratio is the Most Irrational Number!

You should have gotten that

$$\varphi = [1, 1, 1, 1, 1, ...]$$

and that the convergents are ratios of consecutive Fibonacci numbers! Neat, right? Just in case you didn't get that, notice that if $x = [1, 1, 1, ...]$, then

$$x = 1 + \frac{1}{x}.$$

If we solve for x now using the quadratic formula (and know that $x > 0$), then we can see that this is the right continued fraction for φ.

Now let's think about what it means to be "the most irrational." By that, we mean that the rational approximations from the convergents are far away from our irrational number. Think about what it means when one of the a_is is really big, like 100. It means that the last convergent (the one not including the 100) is pretty close! In the next step, we only change by $\frac{1}{100}$, instead of something much bigger, like $\frac{1}{2}$. Concretely,

$$\sqrt{3} = [1, \overline{1, 2}]$$

and

$$1 + \frac{1}{1}$$

is close (relatively speaking) to $\sqrt{3}$ because the next a_i is 2, which is somewhat large (at least it's bigger than 1). The next convergent, namely

$$1 + \cfrac{1}{1 + \cfrac{1}{2}}$$

is pretty close to the previous one.

So to summarize, *big numbers in the continued fraction mean good rational approximations.* So φ, which has all 1's in its continued fraction, has only bad rational approximations!

5. Why the Golden Ratio Occurs in Nature

So now we understand that the golden ratio is very irrational. Great! So why would nature care?

Let's say you're a tree. You have a central trunk and you want to put branches out at regular angles spiraling up your trunk. Now, you want your branches to overlap as little as possible so that they don't block the light for the branches below. So what angle should you choose?

You don't want to choose something like 90°, because you will just have four stacks of branches, and basically only four branches that get to see the light. In other words, you don't want to choose a factor of 360°.

Instead of thinking about angle measurements, let's just think about what fraction of the circle we want to go around each time. What we said just now is that we don't want to go around by $\frac{1}{4}$ of the circle every time. Nor would we want to go around by any other rational number. After a few go rounds (depending on the denominator) our branches will start to stack. So each branch should be an irrational fraction of the circle from the last.

But which irrational number? If we want to be as far away from a rational number as possible, we should choose φ! Of course, turning 1.618... of a circle each time is the same as turning 0.618... each time. So often you might hear the 0.618... number when reading about plant growth.

This same principle applies to flowers putting out petals, growing pinecones, and many other things. The idea is that if you're a flower, you don't want all your petals to overlap. You want to have them all nicely nestled instead of going out on spokes. The following website has a nice simulation. Put in a decimal to rotate each time, and see what kind of pinecone, seed head, petal pattern, etc. you would get.

`http://www.mathisfun.com/numbers/nature-golden-ratio-fibonacci.html`

If you play around with the angles, you'll see that rational numbers (like 0.25 or 0.3333333) create spokes. Irrational numbers (or at least, decimal approximations of irrational numbers, since you can only type decimals into the website) give spiraling arms. "More irrational" irrational numbers will give more tightly packed spirals. The golden ratio looks really good!

A final note: you may have heard that the Fibonacci numbers are the numbers that come up in nature, not just the golden ratio. Well, ratios of Fibonacci numbers approximate the golden ratio! So really, it all makes sense.

For another nice presentation of Fibonacci numbers showing up in nature, check out Vi Hart's videos.

`http://www.vihart.com/blog/doodling-fibonacci-1/`

In conclusion, math is awesome and makes all kinds of sense. Thanks for reading!