RESEARCH STATEMENT

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1. Introduction

My research interests are in analytic number theory, more specifically automorphic forms and multiple Dirichlet series. Currently, my work involves Fourier coefficients of Eisenstein series. By investigating Fourier coefficients of automorphic forms, we gain a deeper understanding of their structure in two senses. We understand the automorphic forms better as functions and also gain insight into how the automorphic forms relate to each other. Dirichlet series, that is, series of the form

$$D(s) = \sum \frac{a(n)}{n^s}$$

are integral to the field of analytic number theory, and they arise naturally as Fourier coefficients of half-integral weight Eisenstein series.

In my thesis I investigate GL_3 Eisenstein series, proving that sums of special values of the minimal parabolic GL_3 Eisenstein series are equal to products of double Dirichlet series. These double Dirichlet series are actually Fourier coefficients of the half-integral weight GL_3 Eisenstein series. The methods involve constructing a multiple Dirichlet series out of the three invariants of the action of upper triangular matrices $P(\mathbb{Z})$ in $SL_3(\mathbb{Z})$ on the space $X^+(\mathbb{Z})$ of positive definite integral ternary quadratic forms. The Dirichlet series has the form

$$\sum_{T \in P(\mathbb{Z})\backslash X^+(\mathbb{Z})} \frac{1}{r_1(T)^{s_1} r_2(T)^{s_2} r_3(T)^{s_3}}.$$

This series is an example of a zeta function associated to a prehomogeneous vector space and originates in the work of Fumihiro Sato (see [12] and [13]). It can be interpreted, on the one hand, as summing values of the full-integral weight Eisenstein series. On the other hand, using Hurwitz zeta functions and genus theory, I show that the multiple Dirichlet series is a product of the double Dirichlet series that arise as Fourier coefficients of the half-integral weight Eisenstein series.

During my postdoctoral tenure, I will extend these ideas further. In particular, I hope to address other GL_3 automorphic forms, such as the maximal parabolic Eisenstein series and GL_3 Maass forms. I plan to show that the relationship between special values of full-integral weight forms and Fourier coefficients of half-integral weight forms continues. In addition, I intend to extend my current work to GL_r Eisenstein series, where the methods should carry over naturally. Also, I will work on constructing multiple Dirichlet series using other prehomogeneous vector spaces, both as a tool for exploring automorphic forms, but also as an end in itself. For example, Datskovsky and Wright [5] use the zeta function attached to the space of binary cubic forms to understand the distribution of cubic discriminants. With my more general construction of the multiple Dirichlet series attached to prehomogeneous vector spaces, I hope to give related applications.

I also have other interests within the theory of automorphic forms and multiple Dirichlet series. Namely, I have a joint paper that proves that Fourier coefficients of a half-integral weight cusp form change sign infinitely often. With the same collaborators, I have an on-going project on counting square discriminants. This paper, which is in preparation, gives a new application of Jeffrey Hoffstein's work on shifted Dirichlet series [7].

2. Joint Work

2.1. **Fourier Coefficients of Half-Integral Weight Cusp Forms.** Waldspurger proved that the normalized Fourier coefficients $a(m)$ of a half-integral weight holomorphic cusp form are, up to some factors,

$$a(m) = \pm\sqrt{L(\frac{1}{2}, f, \chi_m)}$$

when m is square-free and f is the integral weight cusp form corresponding to $\mathfrak{f}$ via the Shimura correspondence. W. Kohnen posed the question: which square root is $a(m)$? In other words, what can we say about the sign of the Fourier coefficients of $\mathfrak{f}$? Joint with Thomas Hulse, Eren Mehmet Kıral, and Chan Ieong Kuan ([9]), I wrote a paper proving that the Fourier coefficients change sign infinitely often.

Our method involves investigating the Dirichlet series

$$M(s) = \sum_{\substack{t\geq 1 \\ t \text{ square-free}}} \frac{a(t)}{t^s}$$

and showing that this series has an analytic continuation to $\Re(s) > \frac{3}{4}$. Then, using a Mellin transform, we get a bound on

$$\sum_t a(t)e^{-t/x}$$

in terms of x. But we also understand the size of

$$\sum_t a(t)^2$$

and see that there is necessarily cancellation in the sum of Fourier coefficients.

2.2. **Ongoing Project: Counting Square Discriminants.** Joint with Thomas Hulse, Eren Mehmet Kıral, Chan Ieong Kuan and Min Lee [8], I prove a result on counting solutions to $b^2-4ac = h$ with $|a|$, $|b|$, and $|c|$ less that X. This paper is currently in preparation, and is likely to be submitted in the spring.

In particular, we prove that if h is a square

$$\#\left\{(a,b,c) \mid b^2 - 4ac = h \text{ and } |a|,|b|,|c| < X\right\} \sim c_1 X \log X + c_2 X + O(X^{\frac{1}{2}})$$

and if h is not a square

$$\#\left\{(a,b,c) \mid b^2 - 4ac = h \text{ and } |a|,|b|,|c| < X\right\} \sim c_3 X + O(X^{\frac{1}{2}}).$$

Our methods involve studying the analytic properties of the shifted double Dirichlet series

$$\sum_{a,c=1}^{\infty} \frac{\tau(4ac+h)}{a^s c^w}$$

where τ is defined by

$$\tau(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n \text{ is a nonzero square} \\ 0 & \text{if } n \text{ is not a square} \end{cases}.$$

This series arises from an inner product of the function $V(z) = y^{\frac{1}{4}}\theta(z)E(z,s)$ against the Poincare series described in [7]. By using the Fourier expansion for the Poincare series, we see that the inner product gives us the shifted Dirichlet series we want to study. On the other hand, we can use the spectral expansion for the Poincare series to obtain asymptotics.

This project refines results of Oh and Shah, and provides a new application for the machinery built up by Hoffstein in [7].

3. Full-integral and Half-integral Weight Forms

3.1. **Background.** Let Λ_d to be the set of Heegner points of discriminant d. In [11], Katok and Sarnak show that for a GL_2 Maass form φ,

$$\sum_{z \in \Lambda_d} \varphi(z) = \sum_{\mathrm{Shim}(F_j)=\varphi} \rho_j(1)\rho_j(d),$$

where the sum on the right side is over all half-integral weight forms that lift to φ under the Shimura correspondence described in [15], and $\rho_j(n)$ denotes the n-th Fourier coefficient of F_j.

Katok and Sarnak's result is a generalization of a well-known result for GL_2 Eisenstein series. Set $E(z,s)$ to be the real analytic GL_2 Eisenstein series defined by

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Im(\gamma z)^s,$$

where $\Gamma = SL_2(\mathbb{Z})$ and $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\}$. Then the following is true.

Proposition 1. *If d a fundamental discriminant,*

$$\sum_{z \in \Lambda_d} E(z,s) = \frac{\Gamma(s/2)}{\pi^s} \zeta(s) L(s, \chi_d) = \frac{\Gamma(s/2)}{\pi^s} \zeta_{\mathbb{Q}(\sqrt{d})}(s),$$

where χ_d is the quadratic character corresponding to the field extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$.

In the proposition, $\zeta(s)$ is the usual Riemann zeta function, $L(s,\chi_d)$ is the quadratic Dirichlet L-series and $\zeta_{\mathbb{Q}(\sqrt{d})}(s)$ is the Dedekind zeta function of $\mathbb{Q}(\sqrt{d})$, defined by

$$\zeta(s) = \sum_{n>0} \frac{1}{n^s} \qquad \text{and} \qquad L(s,\chi_d) = \sum_{n>0} \frac{\chi_d(n)}{n^s} \qquad \text{and} \qquad \zeta_{\mathbb{Q}(\sqrt{d})}(s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_d} \frac{1}{(N\mathfrak{a})^s}.$$

To understand this result in the context of Katok-Sarnak, we recall the definition of the half-integral weight Eisenstein series, $\tilde{E}(z,s)$.

$$\tilde{E}(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} j(\gamma, z)^{-1} \Im(\gamma z)^s,$$

where $\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c \equiv 0 \mod 4 \right\}$. In this definition, the j factor is the factor of automorphy of the theta series, or explicitly,

$$j(\gamma, z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} (cz+d)^{\frac{1}{2}},$$

where $\left(\frac{c}{d}\right)$ is the extension of the Jacobi symbol described by Shimura in [15] and

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \mod 4 \\ i & \text{if } d \equiv 3 \mod 4 \end{cases}.$$

The d-th Fourier coefficient of $\tilde{E}(z,s)$ is essentially $L(s,\chi_d)$ (see, for example, [6]), explaining how Proposition 1 is analogous to the result of Katok and Sarnak.

3.2. **Dissertation Research.** My thesis concentrates on the case of the minimal parabolic Eisenstein series defined in [3]. I prove that sums of special values of the minimal parabolic GL_3 Eisenstein series can be interpreted as products of double Dirichlet series. These double Dirichlet series are known to be the Fourier coefficients of the half-integral weight GL_3 Eisenstein series. Therefore, my thesis can be seen as an extension of the work of Chinta and Offen [4], who prove the result for one particular sum of special values.

However, my methods are very different from Chinta and Offen's. I construct the Shintani zeta function associated to the prehomogeneous vector space X of ternary quadratic forms. This is accomplished by considering the invariants of the action of P on X^+, where P is the minimal parabolic subgroup of $SL_3(\mathbb{Z})$ and X^+ is the space of positive definite forms. If we represent such a form by a symmetric matrix

$$T = \begin{pmatrix} 2p & s & t \\ s & 2q & u \\ t & u & 2r \end{pmatrix}$$

the invariants are

$$\begin{aligned} r_1(T) &= \operatorname{disc} T = r(s^2 - 4pq) + (pu^2 - stu + qt^2) \\ r_2(T) &= s^2 - 4pq \\ r_3(T) &= p. \end{aligned}$$

Then we form the multiple Dirichlet series

$$Z(s_1, s_2, s_3) = \sum_{T \in P \backslash X^+(\mathbb{Z})} \frac{1}{|r_1(T)|^{s_1} |r_2(T)|^{s_2} |r_3(T)|^{s_3}}.$$

On the one hand, by computing explicit coset representatives for the quotient $P \backslash X^+(\mathbb{Z})$ and using genus theory, we see that this series can be seen as a product of double Dirichlet series. On the other hand, by dividing the sum further, we interpret it as a sum of special values of the minimal parabolic GL_3 Eisenstein series.

3.3. **Future Research.** In my postdoctoral work, I will investigate the extension of my results to higher rank groups. One of the motivations here is to provide evidence for Jacquet's conjecture, stated in [10], which relates integrals of $G = GL_r$ cusp forms over orthogonal subgroups of G to Whittaker coefficients on the double cover of G via a relative trace formula. Eisenstein series, because they are very explicitly defined, are easier to work with than general cusp forms and should be the first entry into Jacquet's conjecture.

In particular, I intend to generalize my current results to the case of the maximal parabolic Eisenstein series in the following way:

Problem 1. *Prove that a sum of special values of the maximal parabolic GL_3 Eisenstein series is equal to a product of Fourier coefficients of the corresponding half-integral weight Eisenstein series.*

This problem can be solved using methods developed in my thesis, and I have some preliminary results in this direction. The key to these problems is in choosing an appropriate parabolic subgroup of $SL_3(\mathbb{Z})$ to act on ternary quadratic forms. For the problem of the maximal parabolic Eisenstein series, I will use the action of the subgroup

$$P_{\max} = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix} \in SL_3(\mathbb{Z}) \right\}$$

on ternary quadratic forms, and using the two invariants of this action, I will construct a Dirichlet series in two complex variables. This multiple Dirichlet series will encode the necessary information to get the desired equality.

I also intend to solve the following problems.

Problem 2. *Prove the corresponding result for GL_n Eisenstein series. Namely, show that a sum of values of a GL_n Eisenstein series is equal to a product of Fourier coefficients of the corresponding half-integral weight Eisenstein series.*

Problem 3. *Prove that a sum of special values of a GL_3 Maass cusp form is equal to (up to some factors) a sum of products of Fourier coefficients of corresponding half-integral weight Maass forms.*

To prove these results, I will again use series similar to the one constructed in my thesis. For the GL_n result, I will rely on my experience with GL_2 and GL_3 to generalize. Although there will be more invariants, and therefore more complex variables, a lot of the theory will be analogous.

4. Prehomogeneous Vector Spaces

4.1. **Background.** A prehomogeneous vector space is pair (G, V) of a complex algebraic group G and a complex vector space V such that G acts on V with an open dense orbit. A key fact is that the algebra of relative invariants of such an action is finitely generated. For example, one could take the space of binary quadratic forms with the action of $GL_2(\mathbb{C})$. The algebra of relative invariants is then just generated by the usual discriminant $b^2 - 4ac$.

Recently, Bhargava (in [1] and [2]) has obtained results in counting quartic and quintic number fields. He does this by exploring integer orbits of prehomogeneous vector spaces. He shows that, using some set of operations, one can obtain various other prehomogeneous vector spaces. For example, he constructs a bijection:

$$\{2 \times 2 \times 2 \text{ integral matrices}\} \longleftrightarrow \{(Q_1, Q_2, Q_3) | Q_1 \circ Q_2 \circ Q_3 = I\}$$

where the Q_i are equivalence classes of integral binary quadratic forms and the condition is that their product defined by Gauss composition is the identity. By symmetrizing the matrices (or cubes, as he calls them), he constructs an inclusion of binary cubic forms into the space of cubes:

$$\mathrm{Sym}^3\mathbb{Z}^2 \hookrightarrow \mathbb{Z}^3.$$

Bhargava similarly constructs many other maps between prehomogeneous vector spaces and shows that many prehomogeneous vector spaces arise as some series of transformations of the original cubes.

4.2. **Future Research.** My interest is to look at the Dirichlet series associated to these spaces. Sato and Shintani, in [14], define the zeta function associated to a prehomogenous vector space, known as the Shintani zeta function, using the relative invariant of the group action. They prove some basic results, for example showing that these zeta functions have an analytic continuation and satisfy a functional equation. I would like to study these zeta functions further, generalizing them to series with multiple complex variables and investigating the interplay between Bhargava's multilinear algebraic operations on prehomoegenous vector spaces and the analytic properties of the zeta functions.

In particular, during my postdoctoral tenure, I will study the following problem and answer the following question.

Problem 4. *Construct the multiple Dirichlet series associated to additional prehomogeneous vector spaces arising from the action of a nonreductive group. Prove that these multi-variable series still have an analytic continuation, and show that they satisfy a family of functional equations.*

The results described in the previous section are suggestive of the potential arithmetic applications of studying these multiple Dirichlet series.

Question 1. *What relationship, if any, is there between zeta functions of prehomogeneous vector spaces related by symmetrization, dualization, or any of the other operations described by Bhargava? Do any analytic properties carry over?*

Answers to these questions will provide new context to Bhargava's work and relate his results to the work of Sato and Shintani and are of great interest to me.

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